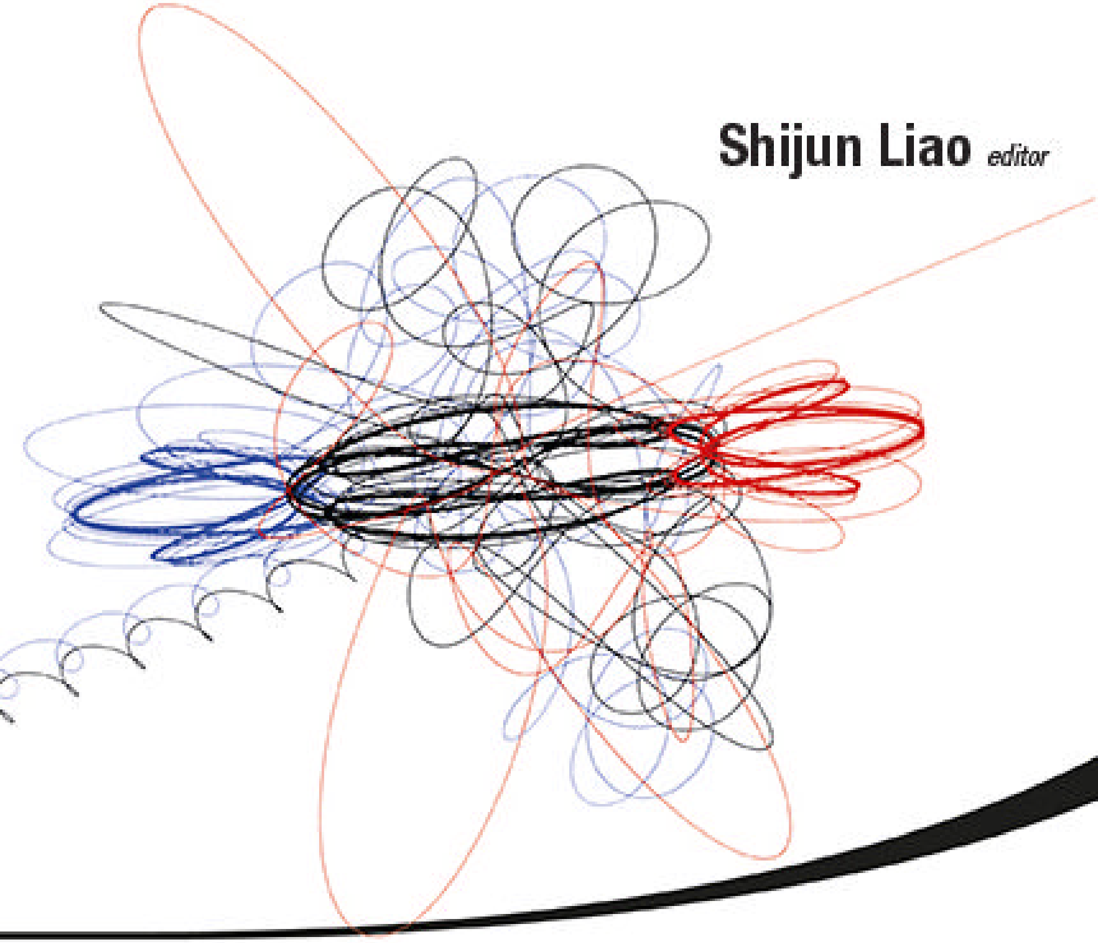
ADVANCES IN THE HOMOTOPY ANALYSIS METHOD

Shijun Liao *editor*



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Preface

The homotopy is a fundamental concept in topology, which can be traced back to Jules Henri Poincaré (1854–1912), a French mathematician. Based on the homotopy, two methods have been developed. One is the homotopy continuation method dating back to 1930s, which is a global convergent numerical method mainly for nonlinear algebraic equations. The other is the homotopy analysis method (HAM) proposed in 1990s by Shijun Liao, the editor of this book, which is an analytic approximation method with guarantee of convergence, mainly for nonlinear differential equations.

Different from perturbation techniques which are strongly dependent upon small/large physical parameters (i.e. perturbation quantities), the HAM has nothing to do with any small/large physical parameters at all. Besides, many analytic approximation methods, such as "Lyapunov artificial small parameter method", "Adomian decomposition method" and so on, are only special cases of the HAM. Unlike other analytic approximation techniques, the HAM provides us great freedom and flexibility to choose equation-type and solution expression of high-order approximation equations. Notice that "the essence of mathematics lies entirely in its freedom", as pointed out by Georg Cantor (1845–1918). Most importantly, different from all of other analytic approximation methods, the HAM provides us a convenient way to guarantee the convergence of approximation series by means of introducing the so-called "convergence-control parameter". In fact, it is the convergence-control parameter that differs the HAM from all other analytic approximation methods. As a result, the HAM is generally valid for various types of equations with high nonlinearity, especially for those without small/large physical parameters.

Since 1992 when the early HAM was first proposed by Liao, the HAM has been developing greatly in theory and applied successfully to numerous types of nonlinear equations in lots of different fields by scientists, researchers, engineers and graduated students in dozens of countries. All of these verify the originality, novelty, validity and generality of the HAM.

So, it is necessary to describe, although briefly, the current advances of the HAM in both theory and applications. This is the first motivation of the book, whose chapters are contributed by the leading researchers in the HAM coming from seven countries.

Any truly new method should give something novel and/or better. In the past 20 years, hundreds of articles related to the HAM were published in various fields, and some new solutions were indeed found by means of the HAM. Thus, it is now the time to suggest some valuable but challenging nonlinear problems to the HAM community. This is the second motivation of the book. Some of these problems are very famous, with a long history. Hopefully, the above-mentioned freedom and flexibility of the HAM might create some novel ideas and inspire brave, enterprising, young researchers with stimulated imagination to attack them with satisfactory results. I personally believe that the applications of the HAM on these famous, challenging problems might not only indicate the great potential of the HAM, but also lead to great modifications of the HAM in theory.

A brief review of the HAM is given in Chapter 1, with some suggested challenging problems. The fascinating "Predictor HAM" and "Spectral HAM" are described in Chapters 2 and 3, respectively. Some interesting theoretical works on the auxiliary linear operator, convergence-control parameter and convergence of approximation series are described in Chapters 4 and 5. An attractive application of the HAM about flows of nanofluid is given in Chapter 6. A charming application of the HAM for time-fractional boundary-value problem is illustrated in Chapter 7. The HAM-based Maple package `NOPH` 1.0.2 (`http://numericaltank.sjtu.edu.cn/NOPH.htm`) for periodic oscillations and limit cycles of nonlinear dynamic systems with various applications is described in Chapter 8. The HAM-based Mathematica package `BVPh` 2.0 (`http://numericaltank.sjtu.edu.cn/BVPh.htm`) for coupled nonlinear ordinary differential equations and its applications are given in Chapter 9. Both of them are easy-to-use, user-friendly, and free available online with user's guide. They can greatly simplify some applications of the HAM.

It is a great pity that it is impossible to describe, even briefly, the whole advances of the HAM in theory and applications in such a book. Here, I would like to express my sincere and truthful acknowledgements to all of the HAM community for their great contributions to the HAM.

Shijun Liao
June 2013, Shanghai

Contents

Chapter 1

Chance and Challenge: A Brief Review of Homotopy Analysis Method

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A brief review of the homotopy analysis method (HAM) and some of its current advances are described. We emphasize that the introduction of the homotopy, a basic concept in topology, is a milestone of the analytic approximation methods, since it is the homotopy which provides us great freedom and flexibility to choose equation type and solution expression of high-order approximation equations. Besides, the so-called "convergence-control parameter" is a milestone of the HAM, too, since it is the convergence-control parameter that provides us a convenient way to guarantee the convergence of solution series and that differs the HAM from all other analytic approximation methods. Relations of the HAM to the homotopy continuation method and other analytic approximation techniques are briefly described. Some interesting but challenging nonlinear problems are suggested to the HAM community. As pointed out by Georg Cantor (1845–1918), "the essence of mathematics lies entirely in its freedom". Hopefully, the above-mentioned freedom and great flexibility of the HAM might create some novel ideas and inspire brave, enterprising, young researchers with stimulated imagination to attack them with satisfactory, better results.

Contents

1.1. Background

Physical experiment, numerical simulation and analytic (approximation) method are three mainstream tools to investigate nonlinear problems. Without doubt, physical experiment is always the basic approach. However, physical experiments are often expensive and time-consuming. Besides, models for physical experiments are often much smaller than the original ones, but mostly it is very hard to satisfy all similarity criterions. By means of numerical methods, nonlinear equations defined in rather complicated domain can be solved. However, it is difficult to gain numerical solutions of nonlinear problems with singularity and multiple solutions or defined in an infinity domain. By means of analytic (approximation) methods, one can investigate nonlinear problems with singularity and multiple solutions in an infinity interval, but equations should be defined in a simple enough domain. So, physical experiments, numerical simulations and analytic (approximation) methods have their inherent advantages and disadvantages. Therefore, each of them is important and useful for us to better understand nonlinear problems in science and engineering.

In general, exact, closed-form solutions of nonlinear equations are hardly obtained. Perturbation techniques [1–4] are widely used to gain analytic approximations of nonlinear equations. Using perturbation methods, many nonlinear equations are successfully solved, and lots of nonlinear phenomena are understood better. Without doubt, perturbation methods make great contribution to the development of nonlinear science. Perturbation methods are mostly based on small (or large) physical parameters, called perturbation quantity. Using small/large physical parameters, perturbation methods transfer a nonlinear equation into an infinite number of subproblems that are mostly linear. Unfortunately, many nonlinear equations do not contain such kind of perturbation quantities at all. More importantly, perturbation approximations often quickly become invalid when the so-called perturbation quantities enlarge. In addition, perturbation techniques are so strongly dependent upon physical small parameters that we have nearly no freedom to choose equation type and solution expression of high-order approximation equations, which are often complicated and thus difficult to solve. Due to these restrictions, perturbation methods are valid mostly for weakly nonlinear problems in general.

On the other side, some non-perturbation methods were proposed long ago. The so-called "Lyapunov's artificial small-parameter method" [5] can trace back to the famous Russian mathematician Lyapunov (1857–1918), who first rewrote a nonlinear equation

$$\mathcal{N}[u(\mathbf{r},t)] = \mathcal{L}_0[u(\mathbf{r},t)] + \mathcal{N}_0[u(\mathbf{r},t)] = f(\mathbf{r},t), \tag{1.1}$$

where $\mathbf{r}$ and t denote the spatial and temporal variables, $u(\mathbf{r},t)$ a unknown function, $f(\mathbf{r},t)$ a known function, $\mathcal{L}_0$ and $\mathcal{N}_0$ are linear and nonlinear operator, respectively, to such a new equation

$$\mathcal{L}_0[u(\mathbf{r},t)] + q\,\mathcal{N}_0[u(\mathbf{r},t)] = f(\mathbf{r},t), \tag{1.2}$$

where q has *no* physical meaning. Then, Lyapunov regarded q as a small parameter to gain perturbation approximations

$$u \approx u_0 + u_1\,q + u_2\,q^2 + u_3\,q^3 + \cdots = u_0 + \sum_{m=1}^{+\infty} u_m\,q^m, \tag{1.3}$$

and finally gained approximation

$$u \approx u_0 + \sum_{m=1}^{+\infty} u_m, \tag{1.4}$$

by setting $q = 1$, where

$$\mathcal{L}_0[u_0(\mathbf{r},t)] = f(\mathbf{r},t), \;\; \mathcal{L}_0[u_1(\mathbf{r},t)] = -\mathcal{N}_0[u_0(\mathbf{r},t)], \cdots \tag{1.5}$$

and so on. It should be emphasized that one has *no* freedom to choose the linear operator $\mathcal{L}_0$ in Lyapunov's artificial small-parameter method: it is exactly the linear part of the whole left-hand side of the original equation $\mathcal{N}[u] = f$, where $\mathcal{N} = \mathcal{L}_0 + \mathcal{N}_0$. Thus, when $\mathcal{L}_0$ is complicated or "singular" (for example, it does not contain the highest derivative), it is difficult (or even impossible) to solve the high-order approximation equation (1.5). Besides, the convergence of the approximation series (1.4) is *not* guaranteed in general. Even so, Lyapunov's excellent work is a milestone of analytic approximation methods, because it is independent of the existence of physical small parameter, even though it first regards q as a "small parameter" but finally enforces it to be 1 that is however *not* "small" strictly from mathematical viewpoints.

The so-called "Adomian decomposition method" (ADM) [6–8] was developed from the 1970s to the 1990s by George Adomian, the chair of the Center for Applied Mathematics at the University of Georgia, USA. Adomian rewrote (1.1) in the form

$$\mathcal{N}[u(\mathbf{r},t)] = \mathcal{L}_A[u(\mathbf{r},t)] + \mathcal{N}_A[u(\mathbf{r},t)] = f(\mathbf{r},t), \tag{1.6}$$

where $\mathcal{L}_A$ often corresponds to the highest derivative of the equation under consideration, $\mathcal{N}_A[u(\mathbf{r},t)]$ gives the left part, respectively. Approximations of the ADM are also given by (1.4), too, where

$$\mathcal{L}_A[u_0(\mathbf{r},t)] = f(\mathbf{r},t), \;\; \mathcal{L}_A[u_m(\mathbf{r},t)] = -A_{m-1}(\mathbf{r},t), \;\; m \geq 1, \tag{1.7}$$

with the so-called Adomial polynomial

$$A_k(\mathbf{r},t) = \frac{1}{k!}\left\{\frac{\partial^k}{\partial q^k}\mathcal{N}_A\left[\sum_{n=0}^{+\infty} u_n(\mathbf{r},t)\; q^n\right]\right\}\Bigg|_{q=0}. \tag{1.8}$$

Since the linear operator $\mathcal{L}_A$ is simply the highest derivative of the considered equation, it is convenient to solve the high-order approximation equations (1.7). This is an advantage of the ADM, compared to "Lyapunov's artificial small-parameter method" [5]. However, the ADM does *not* provides us freedom to choose the linear operator $\mathcal{L}_A$, which is restricted to be related only to the highest derivative. Besides, like "Lyapunov's artificial small-parameter method" [5], the convergence of the approximation series (1.4) given by the ADM is still *not* guaranteed.

Essentially, both of the "Lyapunov's artificial small parameter method" and the "Adomian decomposition method" transfer a nonlinear problem into an infinite number of linear sub-problems, *without* small *physical* parameter. However, they have two fundamental restrictions. First, one has *no* freedom and flexibility to choose the linear operators $\mathcal{L}_0$ or $\mathcal{L}_A$, since $\mathcal{L}_0$ is exactly the linear part of $\mathcal{N}$ and $\mathcal{L}_A$ corresponds to the highest derivative, respectively. Second, there is *no* way to guarantee the convergence of the approximation series (1.4). The second ones is more serious, since divergent approximations are mostly useless. Thus, like perturbation methods, the traditional non-perturbation methods (such as Lyapunov's artificial small parameter method and the ADM) are often valid for weakly nonlinear problems in most cases.

In theory, it is very valuable to develop a new kind of analytic approximation method which should have the following characteristics:

(1) it is *independent* of small physical parameter;
(2) it provides us great *freedom* and *flexibility* to choose the equation-type and solution expression of high-order approximation equations;
(3) it provides us a convenient way to *guarantee* the convergence of approximation series.

One of such kind of analytic approximation methods, namely the "homotopy analysis method" (HAM) [9–17], was developed by Shijun Liao from 1990s to 2010s, together with contributions of many other researchers in theory and applications. The basic ideas of the HAM with its brief history are described below.

1.2. A brief history of the HAM

The basic ideas of "Lyapunov's artificial small-parameter method" can be generalized in the frame of the homotopy, a fundamental concept of topology. For a nonlinear equation

$$\mathcal{N}[u(\mathbf{r},t)] = f(\mathbf{r},t), \tag{1.9}$$

Liao [9] propose the so-called "homotopy analysis method" (HAM) by using the homotopy, a basic concept in topology:

$$(1-q)\mathcal{L}[\varphi(\mathbf{r},t;q) - u_0(\mathbf{r},t)] = c_0\ q\ H(\mathbf{r},t)\ \left\{\mathcal{N}[\varphi(\mathbf{r},t;q)] - f(\mathbf{r},t)\right\}, \tag{1.10}$$

where $\mathcal{L}$ is an auxiliary linear operator with the property $\mathcal{L}[0] = 0$, $\mathcal{N}$ is the nonlinear operator related to the original equation (1.9), $q \in [0,1]$ is the embedding parameter in topology (called the homotopy parameter), $\varphi(\mathbf{r},t;q)$ is the solution of (1.10) for $q \in [0,1]$, $u_0(\mathbf{r},t)$ is an initial guess, $c_0 \neq 0$ is the so-called "convergence-control parameter", and $H(\mathbf{r},t)$ is an auxiliary function that is non-zero almost everywhere, respectively. Note that, in the frame of the homotopy, we have great *freedom* to choose the auxiliary linear operator $\mathcal{L}$, the initial guess $u_0(\mathbf{r},t)$, the auxiliary function $H(\mathbf{r},t)$, and the value of the convergence-control parameter c_0.

When $q = 0$, due to the property $\mathcal{L}[0] = 0$, we have from (1.10) the solution

$$\varphi(\mathbf{r}, t; 0) = u_0(\mathbf{r}, t). \tag{1.11}$$

When $q = 1$, since $c_0 \neq 0$ and $H(\mathbf{r}, t) \neq 0$ almost everywhere, Eq. (1.10) is equivalent to the original nonlinear equation (1.9) so that we have

$$\varphi(\mathbf{r}, t; 1) = u(\mathbf{r}, t), \tag{1.12}$$

where $u(\mathbf{r}, t)$ is the solution of the original equation (1.9). Thus, as the homotopy parameter q increases from 0 to 1, the solution $\varphi(\mathbf{r}, t; q)$ of Eq. (1.10) varies (or deforms) *continuously* from the initial guess $u_0(\mathbf{r}, t)$ to the solution $u(\mathbf{r}, t)$ of the original equation (1.9). For this sake, Eq. (1.10) is called *the zeroth-order deformation equation.*

Here, it must be emphasized once again that we have great freedom and flexibility to choose the auxiliary linear operator $\mathcal{L}$, the auxiliary function $H(\mathbf{r}, t)$, and especially the value of the convergence control parameter c_0 in the zeroth-order deformation equation (1.10). In other words, the solution $\varphi(\mathbf{r}, t; q)$ of the zeroth-order deformation equation (1.10) is also dependent upon all[a] of the auxiliary linear operator $\mathcal{L}$, the auxiliary function $H(\mathbf{r}, t)$ and the convergence-control parameter c_0 as a whole, even though they have *no* physical meanings. This is a key point of the HAM, which we will discuss in details later. Assume that $\mathcal{L}$, $H(\mathbf{r}, t)$ and c_0 are properly chosen so that the solution $\varphi(\mathbf{r}, t; q)$ of the zeroth-order deformation equation (1.10) always exists for $q \in (0, 1)$ and besides it is analytic at $q = 0$, and that the Maclaurin series of $\varphi(\mathbf{r}, t; q)$ with respect to q, i.e.

$$\varphi(\mathbf{r}, t; q) = u_0(\mathbf{r}, t) + \sum_{m=1}^{+\infty} u_m(\mathbf{r}, t)\ q^m \tag{1.13}$$

converges at $q = 1$. Then, due to (1.12), we have the approximation series

$$u(\mathbf{r}, t) = u_0(\mathbf{r}, t) + \sum_{m=1}^{+\infty} u_m(\mathbf{r}, t). \tag{1.14}$$

Substituting the series (1.13) into the zeroth-order deformation equation (1.10) and equating the like-power of q, we have the high-order approximation equations for $u_m(\mathbf{r}, t)$, called the mth-order deformation equation

$$\mathcal{L}[u_m(\mathbf{r}, t) - \chi_m u_{m-1}(\mathbf{r}, t)] = c_0\ H(\mathbf{r}, t)\ R_{m-1}(\mathbf{r}, t), \tag{1.15}$$

[a]More strictly, $\varphi(\mathbf{r}, t; q)$ should be replaced by $\varphi(\mathbf{r}, t; q, \mathcal{L}, \mathrm{H}(\mathbf{r},\mathrm{t}), c_0)$. Only for the sake of simplicity, we use here $\varphi(\mathbf{r}, t; q)$, but should always keep this point in mind.

where

$$R_k(\mathbf{r},t) = \frac{1}{k!}\left\{\frac{\partial^k}{\partial q^k}\left(\mathcal{N}\left[\sum_{n=0}^{+\infty}(\mathbf{r},t)\; q^n\right] - f(\mathbf{r},t)\right)\right\}\Bigg|_{q=0}, \tag{1.16}$$

with the definition

$$\chi_k = \begin{cases} 0, \text{ when } k \leq 1, \\ 1, \text{ when } k \geq 2. \end{cases} \tag{1.17}$$

For various types of nonlinear equations, it is easy and straightforward to use the theorems proved in Chapter 4 of Liao's book [11] to calculate the term $R_k(\mathbf{r},t)$ of the high-order deformation equation (1.15).

It should be emphasized that the HAM provides us great freedom and flexibility to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess u_0. Thus, different from all other analytic methods, the HAM provides us great freedom and flexibility to choose the equation type and solution expression of the high-order deformation equation (1.15) so that its solution can be often gained without great difficulty. Notice that "the essence of mathematics lies entirely in its freedom", as pointed out by Georg Cantor (1845–1918). More importantly, the high-order deformation equation (1.15) contains the convergence-control parameter c_0, and the HAM provides great freedom to choose the value of c_0. Mathematically, it has been proved that the convergence-control parameter c_0 can adjust and control the convergence region and ratio of the approximation series (1.14). For details, please refer to Liao [10, 12, 13] and especially § 5.2 to § 5.4 of his book [11]. So, unlike all other analytic approximation methods, the convergence-control parameter c_0 of the HAM provides us a convenient way to guarantee the convergence of the approximation series (1.14). In fact, it is the convergence-control parameter c_0 that differs the HAM from all other analytic methods.

At the mth-order of approximation, the optimal value of the convergence-control parameter c_0 can be determined by the minimum of residual square of the original governing equation, i.e.

$$\frac{d\mathcal{E}_m}{dc_0} = 0, \tag{1.18}$$

where

$$\mathcal{E}_m = \int_\Omega \left\{\mathcal{N}\left[\sum_{n=0}^{m} u_n(\mathbf{r},t)\right] - f(\mathbf{r},t)\right\}^2 d\Omega. \tag{1.19}$$

Besides, it has been proved by Liao [16] that a homotopy series solution (1.14) must be one of solutions of considered equation, as long as it is convergent. In other words, for an arbitrary convergence-control parameter $c_0 \in \mathbf{R}_c$, where

$$\mathbf{R}_c = \left\{ c_0 : \lim_{m \to +\infty} \mathcal{E}_m(c_0) \to 0 \right\} \tag{1.20}$$

is an interval, the solution series (1.14) is convergent to the true solution of the original equation (1.9). For details, please refer to Liao [16] and Chapter 3 of his book [11].

In summary, the HAM has the following advantages:

(a) it is *independent* of any small/large physical parameters;
(b) it provides us great *freedom* and large *flexibility* to choose equation type and solution expression of linear high-order approximation equations;
(c) it provides us a convenient way to *guarantee* the convergence of approximation series.

In this way, nearly *all* restrictions and limitations of the traditional non-perturbation methods (such as Lyapunov's artificial small parameter method [5], the Adomian decomposition method [6–8], the δ-expansion method [18] and so on) can be overcome by means of the HAM.

Besides, it has been generally proved [10, 12, 13] that the Lyapunov's artificial small parameter method [5], the Adomian decomposition method [6–8] and the δ-expansion method [18] are only special cases of the HAM for some specially chosen auxiliary linear operator $\mathcal{L}$ and convergence-control parameter c_0. Especially, the so-called "homotopy perturbation method" (HPM) [19] proposed by Jihuan He in 1998 (six years later after Liao [9] proposed the early HAM in 1992) was only a special case of the HAM when $c_0 = -1$, and thus has "nothing new except its name" [20]. Some results given by the HPM are divergent even in the whole interval except the given initial/boundary conditions, and thus "it is very important to investigate the convergence of approximation series, otherwise one might get useless results", as pointed out by Liang and Jeffrey [21]. For details, see § 6.2 of Liao's book [11]. Thus, the HAM is more general in theory and widely valid in practice for more of nonlinear problems than other analytic approximation techniques.

In calculus, the famous Euler transform is often used to accelerate convergence of a series or to make a divergent series convergent. It is interesting that one can derive the Euler transform in the frame of the HAM, and give a similar but more general transform (called the generalized Euler transform), as shown in Chapter 5 of Liao's book [11]. This provides us a theoretical cornerstone for the validity and generality of the HAM.

The introduction of the so-called "convergence-control parameter" c_0 in the zeroth-order deformation equation (1.10) is a *milestone* for the HAM. From physical viewpoint, the "convergence-control parameter" c_0 has no physical meanings so that convergent series of solution given by the HAM must be *independent* of c_0. This is indeed true: there exists such a region $\mathbf{R}_c$ that, for arbitrary $c_0 \in \mathbf{R}_c$, the HAM series converges to the true solution of the original equation (1.9), as illustrated by Liao [10, 11]. However, if $c_0 \notin \mathbf{R}_c$, the solution series diverges! So, from a mathematical viewpoint, the "convergence-control parameter" is a key point of the HAM, which provides us a convenient way to guarantee the convergence of the solution series. In fact, it is the so-called "convergence-control parameter" that differs the HAM from all other analytic approximation methods.

The introduction of the basic concept homotopy in topology is also a *milestone* of the analytic approximation methods for nonlinear problems. It is the homotopy that provides us great freedom and large flexibility to choose the auxiliary linear operator $\mathcal{L}$ and initial guess u_0 in the zeroth-order deformation equation (1.10), which determine the equation type and solution expression of the high-order deformation equations (1.15). Besides, it is the homotopy that provides us the freedom to introduce the so-called "convergence-control parameter" c_0 in (1.10), which becomes now a cornerstone of the HAM. Note that it is impossible to introduce such kind of "convergence-control parameter" in the frame of perturbation techniques and the traditional non-perturbation methods (such as Lyapunov's artificial small parameter, Adomian decomposition method and so on).

The freedom on the choice of the auxiliary linear operator $\mathcal{L}$ is so large that the *second-order* nonlinear Gelfand equation can be solved conveniently (with good agreement with numerical results) in the frame of the HAM even by means of a *forth-order* auxiliary linear operator (for two dimensional Gelfand equation) or a *sixth-order* auxiliary linear operator (for three dimensional Gelfand equation), respectively, as illustrated by Liao [14]. Although it is true that the auxiliary linear operator (with the same highest order of derivative as that of considered problem) can be chosen straightforwardly in most cases, such kind of freedom of the HAM should be taken into account sufficiently by the HAM community when necessary, especially for some valuable but challenging problems (some of them are suggested below in § 1.5).

In addition, by means of the above-mentioned freedom of the HAM, the convergence of approximation solution can be greatly accelerated in the frame of the HAM by means of the iteration, the so-called homotopy-Padé technique and so on. For details, please refer to § 2.3.5 to § 2.3.7 of Liao's book [11].

Indeed, "the essence of mathematics lies entirely in its freedom", as pointed out by Georg Cantor (1845–1918).

Such kind of great freedom of the HAM should provide us great possibility to solve some open questions. One of them is described below. The solution of the high-order deformation equation (1.15) can be expressed in the form

$$u_m(\mathbf{r},t) = -\chi_m u_{m-1}(\mathbf{r},t) + \mathcal{L}^{-1}\left[c_0\; H(\mathbf{r},t)\; R_{m-1}(\mathbf{r},t)\right], \tag{1.21}$$

where $\mathcal{L}^{-1}$ is the inverse operator of $\mathcal{L}$. For a few auxiliary linear operator $\mathcal{L}$, its inverse operator is simple. However, in most cases, it is not straightforward to solve the above linear differential equation. Can we directly choose (or *define*) the inverse auxiliary linear operator $\mathcal{L}^{-1}$ so as to solve (1.15) conveniently? This is possible in the frame of the HAM, since in theory the HAM provides us great freedom and large flexibility to choose the auxiliary linear operator $\mathcal{L}$. If successful, it would be rather efficient and convenient to solve the high-order deformation equation (1.15). This is an interesting but open question for the HAM community, which deserves to be studied in details. Note that some interesting problems are suggested in § 1.5.

1.3. Some advances of the HAM

Since 1992 when Liao [9] proposed the early HAM, the HAM has been developing greatly in theory and applications, due to the contributions of many researchers in dozens of countries. Unfortunately, it is impossible to describe all of these advances in details in this brief review, and even in this book. In fact, the HAM has been successfully applied to numerous, various types of nonlinear problems in science, engineering and finance. So, we had to focus on a rather small port of these advances here.

1.3.1. *Generalized zeroth-order deformation equation*

The starting point of the use of the HAM is to construct the so-called zeroth-order deformation equation, which builds a connection (i.e. a continuous mapping/deformation) between a given nonlinear problem and a relatively much simpler linear ones. So, the zeroth-order deformation equation is a base of the HAM.

Given a nonlinear equation, we have great freedom and large flexibility in the frame of the HAM to construct the so-called zeroth-order deformation equation using the concept homotopy in topology. Especially, the convergence-control parameter c_0 plays an important role in the frame of the HAM. So, it is natural to enhance the ability of the so-called "convergence control" by means of introducing more such kind of auxiliary parameters. Due to the above-mentioned freedom and flexibility of the HAM, there are numerous approaches to do so. For example, we can construct such a kind of zeroth-order deformation equation with $K+1$ convergence-control parameters:

$$\begin{aligned}&(1-q)\mathcal{L}\left[\varphi(\mathbf{r},t;q)-u_0(\mathbf{r},t)\right]\\&=\left(\sum_{n=0}^{K} c_n\, q^{n+1}\right)\, H(\mathbf{r},t)\left\{\mathcal{N}[\varphi(\mathbf{r},t;q)]-f(\mathbf{r},t)\right\},\end{aligned} \tag{1.22}$$

where $\varphi(\mathbf{r},t;q)$ is the solution, $\mathcal{N}$ is a nonlinear operator related to an original problem $N[u(\mathbf{r},t)]=f(\mathbf{r},t)$, $q\in[0,1]$ is the homotopy parameter, u_0 is an initial guess, $\mathcal{L}$ is an auxiliary linear operator, $H(\mathbf{r},t)$ is an auxiliary function which is nonzero almost everywhere, and

$$\mathbf{c}=\{c_0,c_1,\cdots,c_K\}$$

is a vector of $(K+1)$ non-zero convergence-control parameters, respectively. Note that, when $K=0$, it gives exactly the zeroth-order deformation equation (1.10).

The corresponding high-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{r},t)-\chi_m u_{m-1}(\mathbf{r},t)]=H(\mathbf{r},t)\sum_{n=0}^{\min\{m-1,K\}} c_n\, R_{m-1-n}(\mathbf{r},t), \tag{1.23}$$

where $R_n(\mathbf{r},t)$ and χ_n are defined by the same formulas (1.16) and (1.17), respectively. When $K=0$, the above high-order deformation equation (1.23) is exactly the same as (1.15). At the mth-order of approximation, the optimal convergence-control parameters are determined by the minimum of the residual square of the original equation, i.e.

$$\frac{d\mathcal{E}_m}{dc_n}=0,\quad 0\le n\le\min\{m-1,K\}, \tag{1.24}$$

where $\mathcal{E}_m$ is defined by (1.19). For details, please refer to Chapter 4 of Liao's book [11]. When $K\to+\infty$, it is exactly the so-called "optimal homotopy asymptotic method" [22]. So, the "optimal homotopy asymptotic