

## Joseph C. Várilly

# An Introduction to Noncommutative Geometry



European Mathematical Society





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# An Introduction to Noncommutative Geometry



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#### Introduction

This book consists of lecture notes for a course given at the EMS Summer School on Noncommutative Geometry and Applications, at Monsaraz and Lisboa, Portugal in September, 1997. These were made available in preprint form on the ArXiv, as physics/9709045, at that time. In updating them for publication, I have kept to the original plan, but have added citations of more recent papers throughout. An extra final chapter summarizes some of the developments in noncommutative geometry in the intervening years.

The course sought to address a mixed audience of students and young researchers, both mathematicians and physicists, and to provide a gateway to noncommutative geometry, as it then stood. It already occupied a wide-ranging area of mathematics, and had received some scrutiny from particle physicists. Shortly thereafter, links to string theory were found, and its interest for theoretical physicists is now indisputable.

Many approaches can be taken to introducing noncommutative geometry. In these lectures, the focus is on the geometry of Riemannian spin manifolds and their noncommutative cousins, which are 'spectral triples' determined by a suitable generalization of the Dirac operator. These 'spin geometries', which are spectral triples with certain extra properties, underlie the noncommutative geometry approach to phenomenological particle models and recent attempts to place gravity and matter fields on the same geometrical footing.

The first two chapters are devoted to commutative geometry; we set up the general framework and then compute a simple example, the two-sphere, in noncommutative terms. The general definition of a spin geometry is then laid out and exemplified with the noncommutative torus. Enough details are given so that one can see clearly that noncommutative geometry is just ordinary geometry, extended by discarding the commutativity assumption on the coordinate algebra. Classification up to equivalence is dealt with briefly in Chapter 7.

Other chapters explore some of the tools of the trade: the noncommutative integral, the role of quantization, and the spectral action functional. Physical models are not treated directly (these were the subject of other lectures at the Summer School), but most of the mathematical issues needed for their understanding are dealt with here. The final chapter is a brief overview of the profusion of new examples and applications of noncommutative spaces and spectral triples.

I wish to thank several people who contributed in no small way to assembling these lecture notes. José M. Gracia-Bondía gave decisive help at many points; and Alejandro Rivero provided constructive criticism. I thank Daniel Kastler, Bruno Iochum, Thomas Schücker and the late Daniel Testard for the opportunity to visit the Centre de Physique Théorique of the CNRS at Marseille, as a prelude to the Summer School; and Piotr M. Hajac for an invitation to teach at the University of Warsaw, when I rewrote the notes for publication. This visit to Katedra Metod Matematycznych Fizyki of UW was supported by European Commission grant MKTD–CT–2004–509794.

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> San Pedro de Montes de Oca April 2006

## Commutative geometry from the noncommutative point of view

The traditional arena of geometry and topology is a *set of points* with some particular structure that, for want of a better name, we call a *space*. Thus, for instance, one studies curves and surfaces as subsets of an ambient Euclidean space. It was recognized early on, however, that even such a fundamental geometrical object as an elliptic curve is best studied not as a set of points (a torus) but rather by examining functions on this set, specifically the doubly periodic meromorphic functions. Weierstrass opened up a new approach to geometry by studying directly the collection of complex functions that satisfy an algebraic addition theorem, and derived the point set as a consequence. In probability theory, the set of outcomes of an experiment forms a measure space, and one may regard events as subsets of outcomes; but most of the information is obtained from 'random variables', i.e., measurable functions on the space of outcomes.

In noncommutative geometry, under the influence of quantum physics, this general idea of replacing sets of points by classes of functions is taken further. In many cases the set is completely determined by an algebra of functions, so one forgets about the set and obtains all information from the functions alone. Also, in many geometrical situations the associated set is very pathological, and a direct examination yields no useful information. The set of orbits of a group action, such as the rotation of a circle by multiples of an irrational angle, is of this type. In such cases, when we examine the matter from the algebraic point of view, we often obtain a perfectly good operator algebra that holds the information we need; however, this algebra is generally not commutative. Thus, we proceed by first discovering how function algebras determine the structure of point sets, and then learning which relevant properties of function algebras do not depend on commutativity.

In a famous paper [94] that has become a cornerstone of noncommutative geometry, Gelfand and Naĭmark in 1943 characterized the involutive algebras of operators by just dropping commutativity from the most natural axiomatization for the algebra of continuous functions on a locally compact Hausdorff space. The starting point for noncommutative geometry that we adopt here is to study ordinary 'commutative' spaces via their algebras of functions, omitting wherever possible any reference to the commutativity of these algebras.

#### 1.1 The Gelfand–Naĭmark cofunctors

The Gelfand–Naĭmark theorem can be thought of as the construction of two contravariant functors (*cofunctors* for short) from the category of locally compact Hausdorff spaces to the category of  $C^*$ -algebras.

The first cofunctor *C* takes a compact space *X* to the *C*\*-algebra *C*(*X*) of continuous complex-valued functions on *X*, and takes a continuous map  $f: X \to Y$  to its transpose  $Cf: C(Y) \to C(X)$ ,  $h \mapsto h \circ f$ . If *X* is only a locally compact space, the corresponding *C*\*-algebra is  $C_0(X)$  whose elements are continuous functions vanishing at infinity, and we require that the continuous maps  $f: X \to Y$  be proper (the preimage of a compact set is compact) in order that  $h \mapsto h \circ f$  take  $C_0(Y)$  into  $C_0(X)$ .

The other cofunctor M goes the other way: it takes a  $C^*$ -algebra A onto its space of *characters*, that is, nonzero homomorphisms  $\mu: A \to \mathbb{C}$ . If A is unital, M(A) is closed in the weak\* topology of the unit ball of the dual space  $A^*$  and hence is compact. If  $\phi: A \to B$  is a unital \*-homomorphism, the cofunctor M takes  $\phi$  to its transpose  $M\phi: M(B) \to M(A), \mu \mapsto \mu \circ \phi$ .

Write  $X^+ := X \uplus \{\infty\}$  for the space X with a point at infinity adjoined (whether X is compact or not), and write  $A^+ := \mathbb{C} \times A$  for the *C*\*-algebra A with an identity adjoined via the rule  $(\lambda, a)(\mu, b) := (\lambda \mu, \lambda b + \mu a + ab)$ , whether A is unital or not; then  $C(X^+) \simeq C_0(X)^+$  as unital *C*\*-algebras. If  $\mu_0 : A^+ \to \mathbb{C}$ ,  $(\lambda, a) \mapsto \lambda$ , then  $M(A) = M(A^+) \setminus \{\mu_0\}$  is locally compact when A is nonunital. Notice that  $M(A)^+$  and  $M(A^+)$  are homeomorphic.

That no information is lost in passing from spaces to  $C^*$ -algebras can be seen as follows. If  $x \in X$ , the *evaluation*  $f \mapsto f(x)$  defines a character  $\varepsilon_x$  in M(C(X)), and the map  $\varepsilon_X \colon X \to M(C(X)), x \mapsto \varepsilon_x$  is a homeomorphism. If  $a \in A$ , its *Gelfand transform*  $\hat{a} \colon M(A) \to \mathbb{C}, \mu \mapsto \mu(a)$  is a continuous function on M(A), and the map  $\mathcal{G} \colon A \to C(M(A)), a \mapsto \hat{a}$  is a \*-isomorphism of  $C^*$ -algebras, that preserves identities if A is unital. These maps are *functorial* (or 'natural') in the sense that the following diagrams commute:



For instance, given a unital \*-homomorphism  $\phi \colon A \to B$ , then for any  $a \in A$  and  $\nu \in M(B)$ , we get

$$((CM\phi \circ \mathcal{G}_A)a)\nu = ((CM\phi)\hat{a})\nu = \hat{a}((M\phi)\nu) = \hat{a}(\nu \circ \phi)$$
$$= \nu(\phi(a)) = \widehat{\phi(a)}(\nu) = ((\mathcal{G}_B \circ \phi)a)\nu,$$

by unpacking the various transpositions.

This 'equivalence of categories' has several consequences. First of all, two commutative  $C^*$ -algebras are isomorphic if and only if their character spaces are homeomorphic. (If  $\phi: A \to B$  and  $\psi: B \to A$  are inverse \*-isomorphisms, then  $M\phi: M(B) \to M(A)$  and  $M\psi: M(A) \to M(B)$  are inverse continuous proper maps.)

Secondly, the group of automorphisms Aut(A) of a commutative  $C^*$ -algebra A is isomorphic to the group of homeomorphisms of its character space. Note that, since A is commutative, there are no nontrivial inner automorphisms in Aut(A).

Thirdly, the topology of X may be specified in terms of algebraic properties of  $C_0(X)$ . For instance, any *ideal* of  $C_0(X)$  is of the form  $C_0(U)$  where  $U \subseteq X$  is an *open subset* (the closed set  $X \setminus U$  being the zero set of this ideal).

If  $Y \subseteq X$  is a *closed* subset of a compact space X, with inclusion map  $j: Y \to X$ , then  $Cj: C(X) \to C(Y)$  is the restriction homomorphism (which is surjective, by Tietze's extension theorem). In general,  $f: Y \to X$  is injective if and only if  $Cf: C(X) \to C(Y)$  is surjective.

We may summarize several properties of the Gelfand–Naĭmark cofunctor with the following dictionary, adapted from [221, p. 24]:

TOPOLOGY	ALGEBRA
locally compact space	C*-algebra
compact space	unital $C^*$ -algebra
compactification	unitization
continuous proper map	*-homomorphism
homeomorphism	automorphism
open subset	ideal
closed subset	quotient algebra
metrizable	separable
Baire measure	positive linear functional

The  $C^*$ -algebra viewpoint also allows one to study the topology of non-Hausdorff spaces, such as arise in probing a continuum where points are unresolved: see the book by Landi on noncommutative spaces [138].

A commutative  $C^*$ -algebra has an abundant supply of characters, one for each point of the associated space. Looking ahead to noncommutative algebras, we can anticipate that characters will be fairly scarce, and we need not bother to search for points. There is, however, one role for points that survives in the noncommutative case: that of zerodimensional elements of a homological skeleton or cell decomposition of a topological space. For that purpose, characters are not needed; we shall require functionals that are only *traces* on the algebra, but are not necessarily multiplicative.

#### **1.2** The $\Gamma$ functor

Continuous functions determine a space's topology, but to do geometry we need at least a differentiable structure. Thus we shall assume from now on that our 'commutative space' is in fact a differential *manifold* M, of dimension n. For simplicity, we shall usually assume that M is *compact*, even though this leaves aside important examples such as Minkowski space. (It turns out that noncommutative geometry has been developed so far almost entirely in the Euclidean signature, where compactness can be seen as a simplifying technical assumption. For the noncompact Euclidean case, see Chapter 9. How to adapt the theory to deal with spaces with indefinite metric is still an open problem, although there are by now several proposals available [133], [166], [203].)

The  $C^*$ -algebra A = C(M) of continuous functions must then be replaced by the algebra  $\mathcal{A} = C^{\infty}(M)$  of *smooth* functions on the manifold M. This is not a  $C^*$ -algebra, and although it is a Fréchet algebra in its natural locally convex topology, our tactic is to work with the dense subalgebra  $\mathcal{A}$  of A in a purely algebraic fashion. We think of  $\mathcal{A}$  as the subspace of 'sufficiently regular' elements of A: see Section 3.4.

A character of  $\mathcal{A}$  is a distribution  $\mu$  on M that is positive, since  $\mu(a^*a) = |\mu(a)|^2 \ge 0$ , and as such is a measure [95] that extends to a character of C(M); hence  $\mathcal{A}$  also determines the point-space M.

To study a given compact manifold M, one uses the category of (complex) vector bundles  $E \xrightarrow{\pi} M$ ; its morphisms are bundle maps  $\tau : E \to E'$  satisfying  $\pi' \circ \tau = \pi$ and so defining fibrewise maps  $\tau_x : E_x \to E'_x$  ( $x \in M$ ) that are required to be linear. Given any vector bundle  $E \xrightarrow{\pi} M$ , write

$$\Gamma(E) := C^{\infty}(M, E)$$

for the space of *smooth sections* of *M*. If  $\tau : E \to E'$  is a bundle map, the composition  $\Gamma \tau : \Gamma(E) \to \Gamma(E')$ ,  $s \mapsto \tau \circ s$  satisfies, for  $a \in \mathcal{A}$ ,  $x \in M$ ,

$$\Gamma\tau(sa)(x) = \tau_x(s(x)a(x)) = \tau_x(s(x))a(x) = (\Gamma\tau(s)a)(x)$$

so  $\Gamma \tau(sa) = \Gamma \tau(s)a$ ; that is,  $\Gamma \tau \colon \Gamma(E) \to \Gamma(E')$  is a morphism of (right) A-modules. One may write either *as* or *sa* to denote the multiplication of a section *s* by a function *a*, so  $\Gamma(E)$  can also be regarded as a left A-module, with appropriate changes of notation.

Vector bundles over M admit operations such as duality, direct sum (i.e., Whitney sum) and tensor product; the  $\Gamma$ -functor carries these to analogous operations on  $\mathcal{A}$ -modules; for instance, if E, E' are vector bundles over M, then

$$\Gamma(E \otimes E') \simeq \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E'),$$

where the right hand side is formed by finite sums  $\sum_j s_j \otimes s'_j$  subject to the relations  $sa \otimes s' - s \otimes as' = 0$ , for  $a \in A$ . One can show that any A-linear map from  $\Gamma(E)$  to  $\Gamma(E')$  is of the form  $\Gamma \tau$  for a unique bundle map  $\tau : E \to E'$ .

It remains to identify the image of the  $\Gamma$ -functor. Note that if  $E = M \times \mathbb{C}^r$  is a trivial bundle, then  $\Gamma(E) = \mathcal{A}^r$  is a *free*  $\mathcal{A}$ -module. Since M is compact, we can find nonnegative functions  $\psi_1, \ldots, \psi_q \in \mathcal{A}$  with  $\psi_1^2 + \cdots + \psi_q^2 = 1$  (a partition of unity) such that E is trivial over the set  $U_j$  where  $\psi_j > 0$ , for each j. If  $f_{ij}: U_i \cap U_j \rightarrow GL(r, \mathbb{C})$  are the transition functions for E, satisfying  $f_{ik}f_{kj} = f_{ij}$  on  $U_i \cap U_j \cap U_k$ , then the functions  $p_{ij} = \psi_i f_{ij}\psi_j$  (defined to be zero outside  $U_i \cap U_j$ ) satisfy  $\sum_k p_{ik}p_{kj} = p_{ij}$ , and so assemble into a  $qr \times qr$  matrix  $p \in M_{qr}(\mathcal{A})$  such that  $p^2 = p$ . A section in  $\Gamma(E)$ , given locally by smooth functions  $s_j: U_j \to \mathbb{C}^r$  such that  $s_i = f_{ij}s_j$  on  $U_i \cap U_j$ , can be regarded as a column vector  $s = (\psi_1s_1, \ldots, \psi_qs_q)^t \in C^{\infty}(M)^{qr}$  satisfying ps = s. In this way, one identifies  $\Gamma(E)$  with  $p\mathcal{A}^{qr}$ .

The *Serre–Swan theorem* [205] says that this is a two-way street: if  $\mathcal{A} = C^{\infty}(M)$ , then any right  $\mathcal{A}$ -module of the form  $p\mathcal{A}^m$ , for an idempotent  $p \in M_m(\mathcal{A})$ , is isomorphic to  $\Gamma(E) = C^{\infty}(M, E)$  for some vector bundle E. The fibre at the point  $\mu \in M = M(\mathcal{A})$  is the vector space  $p\mathcal{A}^m \otimes_{\mathcal{A}} (\mathcal{A}/\ker\mu)$  whose (finite) dimension is the trace of the matrix  $\mu(p) \in M_m(\mathbb{C})$ .

In general, if  $\mathcal{A}$  is any unital algebra, a right  $\mathcal{A}$ -module of the form  $p\mathcal{A}^m$  is called a *finitely generated projective module*. We summarize by saying that  $\Gamma$  is a (covariant) functor from the category of vector bundles over M to the category of finitely generated projective modules over  $C^{\infty}(M)$ . The Serre–Swan theorem gives a recipe to construct an inverse functor going the other way, so that these categories are equivalent. (See the discussion by Brodzki [17], or Chapter 2 of [104], for more details in a modern style.)

What, then, is a *noncommutative vector bundle*? It is simply a finitely generated projective right module  $\mathcal{E}$  for a (not necessarily commutative) algebra  $\mathcal{A}$ , which will generally be a dense subalgebra of a  $C^*$ -algebra  $\mathcal{A}$ .

#### **1.3** Hermitian metrics and spin<sup>c</sup> structures

Any complex vector bundle can be endowed (in many ways) with a Hermitian metric. The conventional practice is to define a positive definite sesquilinear form  $(\cdot | \cdot)_x$  on each fibre  $E_x$  of the bundle, which must 'vary smoothly with x'. The noncommutative point of view is to eliminate x, and what remains is a *pairing*  $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$  on a finitely generated projective right  $\mathcal{A}$ -module with values in the algebra  $\mathcal{A}$  that is  $\mathcal{A}$ -linear in the second variable, hermitian and positive definite. In symbols:

$$(r | s + t) = (r | s) + (r | t),$$
  

$$(r | sa) = (r | s) a,$$
  

$$(r | s) = (s | r)^{*},$$
  

$$(s | s) > 0 \quad \text{for } s \neq 0,$$
  
(1.1)

for  $r, s, t \in \mathcal{E}$ ,  $a \in \mathcal{A}$ . Notice the consequence  $(rb \mid s) = b^* (r \mid s)$  if  $b \in \mathcal{A}$ .

With this structure,  $\mathcal{E}$  is called a *pre-C\*-module* or 'prehilbert module'. More precisely, a pre-*C\**-module over a dense subalgebra  $\mathcal{A}$  of a *C\**-algebra A is a right  $\mathcal{A}$ -module  $\mathcal{E}$  (not necessarily finitely generated or projective) with a pairing  $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$  satisfying (1.1). One can complete it in the norm

$$|||s||| := \sqrt{||(s | s)||}$$

where  $\|\cdot\|$  is the *C*\*-norm of *A*; the resulting Banach space is then a *C*\*-module. In the case  $\mathcal{E} = C^{\infty}(M, E)$ , the completion is the Banach space of continuous sections C(M, E). Indeed, in general this completion is *not* a Hilbert space. For instance, one can take  $\mathcal{E} = \mathcal{A}$  itself, by defining  $(a \mid b) := a^*b$ ; then |||a||| equals the *C*\*-norm ||a||, so the completion is the *C*\*-algebra *A*.

The free A-module  $\mathcal{A}^m$  is a pre- $C^*$ -module in the obvious way, namely  $(r \mid s) := \sum_{j=1}^m r_j^* s_j$ . This column-vector scalar product also works for  $p\mathcal{A}^m$  if  $p = p^2 \in M_m(\mathcal{A})$ , provided that  $p = p^*$  also. If  $q = q^2 \in M_m(\mathcal{A})$ , one can always find a *projector*  $p = p^2 = p^*$  in  $M_m(\mathcal{A})$  that is similar and homotopic to q: see, for example, [104, Thm. 3.8]. (The choice of p selects a particular Hermitian structure on the right module  $q\mathcal{A}^m$ .) Thus we shall always assume from now on that the idempotent p is also selfadjoint.

One can similarly study *left* A-modules. In fact, if  $\mathcal{E}$  is any right A-module, the *conjugate* space  $\overline{\mathcal{E}}$  is a left A-module: by writing  $\overline{\mathcal{E}} = \{\overline{s} : s \in \mathcal{E}\}$ , we can define  $a \overline{s} := (sa^*)^-$ . For  $\mathcal{E} = pA^m$ , we get  $\overline{\mathcal{E}} = \overline{\mathcal{A}}^m p$  where entries of  $\overline{\mathcal{A}}^m$  are to be regarded as 'row vectors'.

**Morita equivalence.** Finitely generated projective A-modules with A-valued pairings play a rôle in noncommutative geometry as *mediating structures* that is partially hidden in commutative geometry: they allow the emergence of new algebras related, but not isomorphic, to A. Consider the 'ket-bra' operators

$$|r\rangle\langle s|: \mathcal{E} \to \mathcal{E}, \quad t \mapsto r \ (s \mid t), \quad \text{for } r, s \in \mathcal{E}.$$
 (1.2)

Composing two ket-bras yields a ket-bra:

$$|r\rangle\langle s|\cdot|t\rangle\langle u| = |r(s|t)\rangle\langle u| = |r\rangle\langle u(t|s)|,$$

so all finite sums of ket-bras form an algebra  $\mathcal{B}$ . Since r(s | ta) = r(s | t) a for  $a \in \mathcal{A}$ , ket-bras act 'on the left' on  $\mathcal{E}$  and commute with the right action of  $\mathcal{A}$ . If  $\mathcal{E} = p\mathcal{A}^m$ , then  $\mathcal{B} = p M_m(\mathcal{A}) p$ . In this way,  $\mathcal{E}$  becomes a ' $\mathcal{B}$ - $\mathcal{A}$ -bimodule'.

If  $\mathcal{A}$  is unital, one can regard  $\mathcal{B}$  as  $\mathcal{E} \otimes_{\mathcal{A}} \overline{\mathcal{E}}$ , by  $|r\rangle\langle s| \leftrightarrow r \otimes \overline{s}$ . On the other hand, we can form  $\overline{\mathcal{E}} \otimes_{\mathcal{B}} \mathcal{E}$ , which is isomorphic to  $\mathcal{A}$  as an  $\mathcal{A}$ -bimodule via  $\overline{r} \otimes s \leftrightarrow (r | s)$ . This is an instance of *Morita equivalence*. In general, we say that two unital algebras  $\mathcal{A}$ ,  $\mathcal{B}$  are *Morita-equivalent* if there is a  $\mathcal{B}$ - $\mathcal{A}$ -bimodule  $\mathcal{E}$  and an  $\mathcal{A}$ - $\mathcal{B}$ -bimodule  $\mathcal{F}$  such that

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B}, \quad \mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A},$$
 (1.3)

as  $\mathcal{B}$ - $\mathcal{B}$ - and  $\mathcal{A}$ - $\mathcal{A}$ -bimodules respectively. With  $\mathcal{E} = \mathcal{A}^m$  and  $\mathcal{F} \simeq \overline{\mathcal{A}}^m$ , we see that any full matrix algebra over  $\mathcal{A}$  is Morita-equivalent to  $\mathcal{A}$ ; nontrivial projectors over  $\mathcal{A}$  offer a host of more 'twisted' examples of algebras that are equivalent to  $\mathcal{A}$  in this sense.

The importance of Morita equivalence of two algebras is that their representations match. More precisely, suppose that there is a Morita equivalence of two algebras  $\mathcal{A}$  and  $\mathcal{B}$ , implemented by a pair of bimodules  $\mathcal{E}$ ,  $\mathcal{F}$  as in (1.3). Then the functors  $\mathcal{H} \mapsto \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$  and  $\mathcal{H}' \mapsto \mathcal{F} \otimes_{\mathcal{B}} \mathcal{H}'$  implement opposing correspondences between representation spaces of  $\mathcal{A}$  and  $\mathcal{B}$ .

*Moral*: if we study an algebra  $\mathcal{A}$  only through its representations, we must simultaneously study the various algebras Morita-equivalent to  $\mathcal{A}$ . In particular, we package together the commutative algebra  $C^{\infty}(M)$  and the noncommutative algebra  $M_n(C^{\infty}(M))$  for the purpose of doing geometry.

In the category of  $C^*$ -algebras (with or without unit element), one replaces finitely generated projective modules by arbitrary  $C^*$ -modules and obtains a much richer theory; see, for instance, [137], [181] and especially [175]. The notion analogous to (1.3) is called 'strong Morita equivalence'. In particular, let us note that two  $C^*$ -algebras A and B are strongly Morita equivalent whenever  $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$ , where  $\mathcal{K}$  is the elementary  $C^*$ -algebra of compact operators on a separable, infinite-dimensional Hilbert space [19].

**Spin**<sup>*c*</sup> **structures.** Returning once more to ordinary manifolds, suppose that *M* is an *n*-dimensional orientable Riemannian manifold with a metric *g* on its tangent bundle *T M*. We build a Clifford algebra bundle  $\mathbb{C}\ell'(M) \to M$  whose fibres are full matrix algebras (over  $\mathbb{C}$ ), as follows. If *n* is even, n = 2m, then  $\mathbb{C}\ell'_x(M) := \mathbb{C}\ell(T_xM, g_x) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2^m}(\mathbb{C})$  is the complexified Clifford algebra over the tangent space  $T_xM$ . If *n* is odd, n = 2m + 1, the analogous fibre splits as  $M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$ , so we take only the *even* part of the complexified Clifford algebra:  $\mathbb{C}\ell'_x(M) := \mathbb{C}\ell^{\text{even}}(T_xM) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2^m}(\mathbb{C})$ . The price we pay for this choice is that we lose the  $\mathbb{Z}_2$ -grading of the Clifford algebra bundle in the odd-dimensional case.

What we gain is that in all cases, the bundle  $\mathbb{C}\ell'(M) \to M$  is a locally trivial field of (finite-dimensional) elementary  $C^*$ -algebras. Such a field is classified, up to equivalence, by a third-degree Čech cohomology class  $\delta(\mathbb{C}\ell'(M)) \in H^3(M, \mathbb{Z})$  called the Dixmier–Douady class [67], [175]. Locally, one finds trivial bundles with fibres  $S_x$ such that  $\mathbb{C}\ell'_x(M) \simeq \operatorname{End}(S_x)$ ; the class  $\delta(\mathbb{C}\ell'(M))$  is precisely the obstruction to patching them together (there is no obstruction to the existence of the algebra bundle  $\mathbb{C}\ell'(M)$ ). It was shown by Plymen [171] that  $\delta(\mathbb{C}\ell'(M)) = W_3(TM)$ , the integral class that is the obstruction to the existence of a *spin<sup>c</sup> structure* in the conventional sense of a lifting of the structure group of TM from SO(*n*) to Spin<sup>c</sup>(*n*): see [149, Appendix D] for more information on  $W_3(TM)$ .

Thus *M* admits spin<sup>*c*</sup> structures if and only if  $\delta(\mathbb{C}\ell'(M)) = 0$ . But in the Dixmier– Douady theory,  $\delta(\mathbb{C}\ell'(M))$  is the obstruction to constructing a *B*-*A*-bimodule *S* that implements a (strong) Morita equivalence between the  $C^*$ -algebras  $A = C_0(M)$  and  $B = C_0(M, \mathbb{C}\ell'(M))$ . Let us paraphrase Plymen's redefinition of a spin<sup>c</sup> structure, in the spirit of noncommutative geometry.

**Definition 1.** Let *M* be a Riemannian manifold,  $A = C_0(M)$  and  $B = C_0(M, \mathbb{C}\ell'(M))$ . We say that the tangent bundle *T M admits a spin<sup>c</sup> structure* if and only if it is orientable and  $\delta(\mathbb{C}\ell'(M)) = 0$ . In that case, a *spin<sup>c</sup> structure* on *T M* is a pair ( $\varepsilon$ ,  $\delta$ ) where  $\varepsilon$  is an orientation on *T M* and  $\delta$  is a *B*-*A*-equivalence bimodule.

Following an earlier terminology introduced by Atiyah, Bott and Shapiro [4] in their seminal paper on Clifford modules, the pair ( $\varepsilon$ ,  $\vartheta$ ) is also called a *K*-orientation on *M*. Notice that *K*-orientability demands more than mere orientability in the cohomological sense. In any case, from now on we consider only orientable manifolds *M* with a fixed orientation  $\varepsilon$ , so that *K*-orientability amounts to the existence of  $\vartheta$ . We note in passing that Plymen's approach recovers earlier work of Karrer on Clifford actions [123], [191].

What is this equivalence bimodule \$? By the Serre–Swan theorem, it is of the form  $\Gamma(S)$  for some complex vector bundle  $S \to M$  that also carries an irreducible left action of the Clifford algebra bundle  $\mathbb{C}\ell'(M)$ . This is the *spinor bundle* whose existence displays the spin<sup>c</sup> structure in the conventional picture. We call  $\Gamma(S) = C^{\infty}(M, S)$  the *spinor module*; it is an irreducible Clifford module in the terminology of [4], and has rank  $2^m$  over  $C^{\infty}(M)$  if n = 2m or 2m + 1.

Another matter is how to fit into this picture *spin structures* on M (liftings of the structure group of TM from SO(n) to Spin(n) rather than Spin<sup>c</sup>(n)). These are distinguished by the availability of a *conjugation operator J* on the spinors (which is antilinear); we shall take up this matter in Chapter 3.

*To summarize*: the language of bimodules and Morita equivalence gives us direct access to noncommutative (or commutative) vector bundles without invoking the concept of a 'principal bundle'. The concept of a noncommutative principal bundle is certainly available – see, for instance, [107], [110], [143] and especially [7] – but here we leave this matter aside.

#### 1.4 The Dirac operator and the distance formula

As soon as a spinor module makes its appearance, one can introduce the *Dirac operator*. This is a selfadjoint first-order differential operator  $\not{D}$  defined on the space  $\mathcal{H} := L^2(M, S)$  of square-integrable spinors, whose domain includes the smooth spinors  $\mathscr{S} = C^{\infty}(M, S)$ . If M is even-dimensional, there is a  $\mathbb{Z}_2$ -grading  $\mathscr{S} = \mathscr{S}^+ \oplus \mathscr{S}^-$  arising from the grading of the Clifford algebra bundle  $\Gamma(\mathbb{C}\ell(M))$ , which in turn induces a grading of the Hilbert space  $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ ; let us call the grading operator  $\Gamma$ , so that  $\Gamma^2 = 1$  and  $\mathcal{H}^{\pm}$  are its ( $\pm 1$ )-eigenspaces. The Dirac operator is obtained by composing the natural *covariant derivative* on the modules  $\mathscr{S}^{\pm}$  (or just on  $\mathscr{S}$  in the odd-dimensional case) with the *Clifford multiplication by* 1-*forms* that reverses the grading.

We repeat that in more detail. The Riemannian metric  $g = [g_{ij}]$  defines isomorphisms  $T_x M \simeq T_x^* M$  and induces a metric  $g^{-1} = [g^{ij}]$  on the cotangent bundle  $T^*M$ . Via this isomorphism, we can redefine the Clifford algebra as the bundle with fibres  $\mathbb{C}\ell'_x(M) := \mathbb{C}\ell(T_x^*M, g_x^{-1}) \otimes_{\mathbb{R}} \mathbb{C}$  (replacing  $\mathbb{C}\ell$  by  $\mathbb{C}\ell^{\text{even}}$  when dim M is odd). Let  $\mathcal{A}^1(M) := \Gamma(T^*M)$  be the  $\mathcal{A}$ -module of 1-forms on M. The spinor module  $\mathscr{S}$  is then a  $\mathcal{B}$ - $\mathcal{A}$ -bimodule on which the algebra  $\mathcal{B} = \Gamma(\mathbb{C}\ell'(M))$  acts irreducibly and obeys the anticommutation rule

$$\{\gamma(\alpha), \gamma(\beta)\} = 2g^{-1}(\alpha, \beta) = 2g^{ij}\alpha_i\beta_j \quad \text{for } \alpha, \beta \in \mathcal{A}^1(M).$$
(1.4)

Here  $\gamma : \mathcal{A}^1(M) \to \mathcal{B}$  denotes the action of  $\mathcal{A}^1(M)$  on  $\mathcal{H}$ .

The metric  $g^{-1}$  on  $T^*M$  gives rise to a canonical *Levi-Civita connection*  $\nabla^g : \mathcal{A}^1(M) \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^1(M)$  that, as well as obeying the Leibniz rule

$$\nabla^g(\omega a) = (\nabla^g \omega) \, a + \omega \otimes da,$$

preserves the metric and is torsion-free. A 'spin<sup>*c*</sup> connection' is then a linear operator  $\nabla^S \colon \Gamma(S) \to \Gamma(S) \otimes_{\mathcal{A}} \mathcal{A}^1(M)$  satisfying two Leibniz rules, one for the right action of  $\mathcal{A}$  and the other, involving the Levi-Civita connection, for the left action of the Clifford algebra:

$$\nabla^{S}(\psi a) = \nabla^{S}(\psi) a + \psi \otimes da,$$
  

$$\nabla^{S}(\gamma(\omega)\psi) = \gamma(\nabla^{g}\omega)\psi + \gamma(\omega)\nabla^{S}\psi,$$
(1.5)

for  $a \in \mathcal{A}$ ,  $\omega \in \mathcal{A}^1(M)$ ,  $\psi \in \mathcal{S}$ . In the presence of a spin structure with conjugation operator J, we say  $\nabla^S$  is the *spin connection* if it also commutes with J; this spin connection is unique [104, Sec. 9.3].

Once the spin connection is found, we define the Dirac operator as the composition  $(-i)\gamma \circ \nabla^S$ ; more precisely, the local expression

$$\mathbf{D} := -i\,\gamma(dx^j)\,\nabla^S_{\partial/\partial x^j} \tag{1.6}$$

is independent of the coordinates and defines  $\not{D}$  on the domain  $\mathscr{S} \subset \mathscr{H}$ . The factor (-i) is needed for  $\not{D}$  to be selfadjoint instead of skewadjoint, when we adopt the positive-definite (Euclidean) convention for the Clifford relations (1.4). One can check that this operator is symmetric; it extends to an unbounded selfadjoint operator on  $\mathscr{H}$ , also called  $\not{D}$ . Since M is compact, the latter  $\not{D}$  is a Fredholm operator and its kernel is finite-dimensional. On the orthogonal complement of (ker  $\not{D}$ ) we may define  $\not{D}^{-1}$ , which is a compact operator.

**The distance formula.** The Dirac operator may be characterized more simply by its Leibniz rule. Since the algebra  $\mathcal{A}$  is represented on the spinor space  $\mathcal{H}$  by multiplication

operators, we may form  $\mathcal{D}(a\psi)$ , for  $a \in \mathcal{A}$  and  $\psi \in \mathcal{H}$ . It is an easy consequence of (1.5) and (1.6) that

$$\mathcal{D}(a\psi) = -i\gamma(da)\psi + a\mathcal{D}\psi. \tag{1.7}$$

This is the rule that we need to keep in mind. We can equivalently write it as

$$[\not\!\!D, a] = -i \, \gamma(da).$$

In particular, since *a* is smooth and *M* is compact, the operator  $\|[\mathcal{D}, a]\|$  is *bounded*, and its norm is simply the sup-norm  $\|da\|_{\infty}$  of the differential *da*. This also equals the *Lipschitz seminorm* of *a*, defined as

$$||a||_{\operatorname{Lip}} := \sup_{p \neq q} \frac{|a(p) - a(q)|}{d(p,q)},$$

where d(p, q) is the geodesic distance between the points p and q of the Riemannian manifold M. This might seem to be an unwelcome return to the use of points in geometry; but in fact this simple observation (by Connes) led to one of the great coups of noncommutative geometry [36]. One can simply stand the previous formula on its head:

$$d(p,q) = \sup\{|a(p) - a(q)| : a \in C(M), ||a||_{\text{Lip}} \le 1\},$$
  
= sup{|( $\hat{p} - \hat{q}$ )(a)| :  $a \in C(M), ||[\not{D}, a]|| \le 1\},$  (1.8)

and one discovers that the metric on the space of characters M = M(A) is entirely determined by the Dirac operator.

This is, of course, just a tautology in commutative geometry; but it opens the way forward, since it shows that what one must carry over to the noncommutative case is precisely this operator, or a suitable analogue. One still must deal with the scarcity of characters for noncommutative algebras. The lesson that (1.8) teaches [39] is that the *length element ds* is in some sense inversely proportional to D.

The ingredients for a reformulation of commutative geometry in algebraic terms are almost in place. We list them briefly: an algebra  $\mathcal{A}$ ; a representation space  $\mathcal{H}$  for  $\mathcal{A}$ ; and a selfadjoint operator  $\mathcal{D}$  on  $\mathcal{H}$ . Additionally, a conjugation operator J, still to be discussed; and, in even-dimensional cases, a  $\mathbb{Z}_2$ -grading operator  $\Gamma$  on  $\mathcal{H}$ . This package of four or five terms is called a *real spectral triple* or a *real K-cycle* or, more simply, a *spin geometry*. Our task will be to study, to exemplify, and where possible, to parametrize these geometries.

### Spectral triples on the Riemann sphere

We now undertake the construction of some spectral triples ( $\mathcal{A}, \mathcal{H}, D; \Gamma, J$ ) for a very familiar commutative manifold, the Riemann sphere  $\mathbb{S}^2$ . This is an even-dimensional Riemannian spin manifold, indeed it is the simplest nontrivial representative of that class. Nevertheless, the associated spectral triples are not completely transparent, and their construction is very instructive.

The sphere  $\mathbb{S}^2$  can also be regarded as the complex projective line  $\mathbb{C}P^1$ , or as the compactified plane  $\mathbb{C}_{\infty} = \mathbb{C} \cup \{\infty\}$ . As such, it is described by two charts,  $U_N$  and  $U_S$ , that omit respectively the north and south poles, with the respective local complex coordinates

$$z = e^{-i\phi} \cot \frac{\theta}{2}, \quad \zeta = e^{i\phi} \tan \frac{\theta}{2},$$

related by  $\zeta = 1/z$  on the overlap  $U_N \cap U_S$ . We write  $q(z) := 1 + z\overline{z}$  for convenience. The Riemannian metric g and the area form  $\Omega$  are given by

$$g = d\theta^{2} + \sin^{2}\theta \, d\phi^{2} = 4q(z)^{-2} \, dz \, d\overline{z} = 4q(\zeta)^{-2} \, d\zeta \, d\overline{\zeta},$$
  
$$\Omega = \sin\theta \, d\theta \wedge d\phi = 2i \, q(z)^{-2} \, dz \wedge d\overline{z} = 2i \, q(\zeta)^{-2} \, d\zeta \wedge d\overline{\zeta}.$$

#### 2.1 Line bundles and the spinor bundle

Hermitian line bundles over  $\mathbb{S}^2$  correspond to finitely generated projective modules over  $\mathcal{A} := C^{\infty}(\mathbb{S}^2)$ , of 'rank one'; these are of the form  $\mathcal{E} = p\mathcal{A}^n$  where  $p = p^2 = p^* \in M_n(\mathcal{A})$  is a projector of constant rank 1. (Equivalently,  $\mathcal{E}$  is of rank one if End<sub> $\mathcal{A}$ </sub>( $\mathcal{E}$ )  $\simeq \mathcal{A}$ .) It turns out that it is enough to consider the case  $p \in M_2(\mathcal{A})$ . We follow the treatment of Mignaco *et al.* [160]; see also [104, Sec. 2.6].

Using Pauli matrices  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma_3$ , we may write any projector in  $M_2(\mathcal{A})$  as

$$p = \frac{1}{2} \begin{pmatrix} 1+n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{pmatrix} = \frac{1}{2} (1 + \vec{n} \cdot \vec{\sigma})$$

where  $\vec{n}$  is a smooth function from  $\mathbb{S}^2$  to  $\mathbb{S}^2$ . Any homotopy between two such functions yields a homotopy between the corresponding projectors p and q; and one can then construct a unitary element  $u \in M_4(\mathcal{A})$  such that  $u(p \oplus 0)u^{-1} = q \oplus 0$ . Thus inequivalent finitely generated projective modules are classified by the homotopy group  $\pi_2(\mathbb{S}^2) = \mathbb{Z}$ , the corresponding integer m being the *degree* of the map  $\vec{n}$ . If  $f(z) = (n_1 - in_2)/(1 - n_3)$  is the corresponding map on  $\mathbb{C}_{\infty}$  after stereographic projection,