An Introduction to Semiclassical and Microlocal Analysis

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FOREWORD

The following lecture notes correspond to a course taught for several years, first at the University of Paris-Nord (France) and then at the University of Bologna (Italy). They are mainly addressed to non specialists in the subject, and their purpose is to present in a pedagogical way most of the techniques used in the microlocal treatment of semiclassical problems coming from quantum physics. Both the standard C^∞ pseudodifferential calculus and the analytic microlocal analysis are developed, in a context which remains intentionally global so that only the relevant difficulties of the theory are encountered. The main originality lies in the fact that we derive all the main features of analytic microlocal analysis from a single a priori estimate, which turns out to be elementary once the C^∞ pseudodifferential calculus is established.

Various detailed exercises are given at the end of the main chapters, most of them being easily solvable by students. Beside illustrating the main results of the lecture, their aim is also to introduce the reader to various further developments of the theory, such as the functional calculus of pseudodifferential operators, properties of the analytic wave front set, Gevrey classes, the use of coherent states, the notion of semiclassical measures, WKB constructions, etc ... Applications to the study of the Schrödinger operator are also discussed during the text, so that they may help the understanding of new notions or general results where they appear by replacing them in the context of quantum mechanics. We invite the reader who wishes to find these applications easily to refer to the index which we have tried to make as complete as possible.

The prerequisities are essentially reduced to the basic notions of the theory of distributions.

CONTENTS

Chapter 1: Introduction
- 1.1 A Short Review of Classical Mechanics
- 1.2 Basic Notions of Quantum Mechanics
- 1.3 Semiclassical Analysis
- 1.4 About History

Chapter 2: Semiclassical Pseudodifferential Calculus
- 2.1 Motivations and Notations
- 2.2 Spaces of Symbols
- 2.3 Semiclassical Expansions of Symbols
- 2.4 Oscillatory Integrals
- 2.5 Pseudodifferential Operators
- 2.6 Composition
- 2.7 Change of Quantization - Symbolic Calculus
- 2.8 L^2-Continuity
- 2.9 A First Application: The Frequency Set
- 2.10 Exercises and Problems

Chapter 3: Microlocalization
- 3.1 The Global F.B.I. Transform
- 3.2 Microsupport
- 3.3 Action of the F.B.I. Transform on Pseudodifferential Operators
- 3.4 Action of the F.B.I. Transform on F.I.O.'s
- 3.5 Microlocal Exponential Estimates
- 3.6 Exercises and Problems

Chapter 4: Applications to the Solutions of Analytic Linear P.D.E.'s
- 4.1 Projection of the Microsupport
- 4.2 Characteristic Set and Microsupport
- 4.3 Propagation of the Microsupport
- 4.4 Microhyperbolicity
- 4.5 Boundary Value Problems
- 4.6 Exercises and Problems

Chapter 5: Complements: Symplectic Aspects

5.1 The Canonical Symplectic Form on $\mathbf{R}^{2n}$
5.2 Symplectic Transformations
5.3 Hamiltonian Field
5.4 Lagrangian Submanifolds
5.5 Applications to P.D.E.'s
5.5.1 Solutions of Eikonal Equations
5.5.2 Canonical Transformations of Pseudodifferential Operators

Chapter 6: Appendix: List of Formulæ
6.1 Change of Quantization
6.2 Composition Formulæ
6.3 Symbol of a Commutator
6.4 Functional Calculus
6.5 Fourier-Bros-Iagolnitzer Transform
6.6 Microlocal Exponential Estimates
6.7 Agmon Estimates

Bibliography

Index

List of Notations

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Chapter 1

Introduction

1.1 A Short Review of Classical Mechanics

In this section and in the following one our aim is only to give a slight flavor of the process which has led the physicists to change completely their conception of reality by passing from classical mechanics to quantum mechanics. For a much more detailed presentation of this, we refer the reader to the very complete book [Mes] by A. Messiah. Interesting information can also be found in the book [Ro] by D. Robert (which is actually closer to our point of view), and in the classical book [LaLi] by L.D. Landau and E.M. Lifshitz. (Indeed, a large literature - in any language - exists on the foundations of quantum mechanics, and many more references can be found than we can give here.)

In classical mechanics, a particle of mass m is represented by its *position* at time t, that is by a function $t \mapsto x(t) \in \mathbf{R}^3$. If this particle is submitted to a conservative force field $F = -\nabla V$, then its movement is ruled by Newton's fundamental law:

$$F = m\ddot{x}(t) \tag{1.1.1}$$

or equivalently

$$\ddot{x}(t) = -\frac{1}{m}\nabla V\left(x(t)\right) \tag{1.1.2}$$

where the dots stand for differentiations with respect to t. If we set $\xi(t) = m\dot{x}(t)$ (the so-called *momentum* or *impulse* of the particle), then (1.1.2) can

be rewritten as

$$\begin{cases} \dot{\xi}(t) = -\nabla V\,(x(t)) \\ \dot{x}(t) = \dfrac{1}{m}\xi(t) \end{cases} \tag{1.1.3}$$

which is called the system of Hamilton's equations. The curve $t \mapsto (x(t), \xi(t))$ is then called the *phase space trajectory* or *classical trajectory* of the particle, and lies in $\mathbf{R}^3 \times \mathbf{R}^3$ which should be viewed as the product of the space of positions and momenta.

The *(total) energy* of the particle is defined by

$$E = \frac{1}{2m}\xi(t)^2 + V(x(t)). \tag{1.1.4}$$

The main feature of the energy is that it is independent of t: indeed one has

$$\frac{d}{dt}\left(\frac{1}{2m}\xi(t)^2 + V(x(t))\right) = m\ddot{x}(t).\dot{x}(t) + \nabla V(x(t)).\dot{x}(t) = 0$$

where the last equality is a direct consequence of (1.1.2). Note that the energy of the particle is just the value at $(x(t), \xi(t))$ of the so-called *energy observable* $\dfrac{1}{2m}\xi^2 + V(x)$.

More generally, one calls a *classical observable* any real smooth function $a = a(x, \xi)$ defined on the phase space $\mathbf{R}^3 \times \mathbf{R}^3$: its value at the point $(x(t), \xi(t))$ gives information about the particle at time t. In particular, any physical experiment concerning the particle should lead to quantities which can be described by such values of classical observables.

However, it turns out that in several experiments (such as the photoelectric effect and the diffraction of particles: see [Mes, Ro]), properties in contradiction with this classical model of mechanics have appeared. The most well known are the facts that the energy of a particle can take values only in a discrete subset of $\mathbf{R}$, and that one cannot know at the same time the precise values of both the position and the momentum of the particle: this is the famous *Heisenberg uncertainty principle*, which asserts that the errors Δx and $\Delta\xi$ made in a measurement of the position and the momentum always satisfy

$$\Delta x.\Delta\xi \geq \frac{h}{4\pi} \tag{1.1.5}$$

where h is the Planck constant whose value is approximately $6.6 \times 10^{-34}\ J/s$.

These observations have led physicists to believe in a kind of double nature of elementary particles, both wave-like and corpuscular. After a first attempt in 1923 by De Broglie [DeB] to include such observations in a mathematical model (the so-called *Matter Waves* which generalizes to matter the double aspect - wave-like and corpuscular - previously observed for the light), it is now commonly admitted that a very general and acceptable model is given by another theory of matter: Quantum Mechanics introduced in two equivalent forms around 1925: by M. Born, W. Heisenberg and P. Jordan (the *Matrix Mechanics*: see e.g. [BoHeJo]), and by E. Schrödinger (the *Wave Mechanics*: see [Schr1, Schr2]). About the links between these two presentations, one may consult e.g. [VdW] and references therein.

1.2 Basic Notions of Quantum Mechanics

In quantum mechanics, a particle is described by a function $\mathbf{R}\times\mathbf{R}^3 \ni (t,x) \mapsto \psi(t,x) \in \mathbf{C}$ which is called the *wave function* of the particle. The wave function must be such that for any $t \in \mathbf{R}$, the function $\psi_t : \ x \mapsto \psi(t,x)$ belongs to $L^2(\mathbf{R}^3)$, and $\|\psi_t\|_{L^2(\mathbf{R}^3)} = 1$. The function ψ_t is called the *state* of the particle at time t.

The natural interpretation attached to $\psi(t,x)$ is to view $|\psi(t,x)|^2$ as a density of probability: it describes the probability of presence of the particle at the point x at time t.

The *average position* of the particle at time t is defined in a natural way as the quantity

$$\langle x\rangle_{\psi_t} := \langle x\psi_t, \psi_t\rangle_{L^2(\mathbf{R}^3)} = \Big(\langle x_j\psi_t, \psi_t\rangle_{L^2(\mathbf{R}^3)}\Big)_{j=1,2,3}. \tag{1.2.1}$$

An *average impulse* can also be defined, but its understanding requires an analogy with a *plane wave* given in Optics by a function of the type

$$\varphi(t,x) = Ae^{i(k.x-\omega t)} \tag{1.2.2}$$

where $\nu := \dfrac{\omega}{2\pi}$ represents the frequency, and $k \in \mathbf{R}^3$ is called the *wave vector*: the wave propagates along the direction of k, in the sense that φ is independent of x on any plane $\{x.k = \text{constant}\}$. As a consequence, it is natural that any acceptable definition of the impulse of such a wave must satisfy:

$$\xi = \alpha k$$

for some positive constant α. Actually, the so-called De Broglie relation (derived from considerations on free wave-packets: see e.g. [Mes]) gives

$$\xi = \hbar k \tag{1.2.3}$$

where $\hbar := \dfrac{h}{2\pi}$ is the reduced Planck constant. Using (1.2.2) and (1.2.3) we get in particular:

$$\xi = \frac{\hbar}{i}\left[\nabla_x \varphi(t,x)\right]\overline{\varphi(t,x)}/|A|^2. \tag{1.2.4}$$

It is precisely relation (1.2.4) which provides a way to define by analogy the average impulse of the quantum particle described by $\psi(t,x)$. Viewing $|A|^2$ in (1.2.4) as a normalization factor, one sets:

$$\langle \xi \rangle_{\psi_t} := \langle \frac{\hbar}{i}\nabla\psi_t, \psi_t \rangle_{L^2(\mathbf{R}^3)} = \left(\langle \frac{\hbar}{i}\frac{\partial \psi_t}{\partial x_j}, \psi_t \rangle_{L^2(\mathbf{R}^3)} \right)_{j=1,2,3} \tag{1.2.5}$$

whenever it is defined (e.g. if $\psi_t \in H^1(\mathbf{R}^3)$, the usual Sobolev space).

At this point it is useful to make a connection (which will be essential in the sequels) between (1.2.5) and an equivalent way of writing $\langle \xi \rangle_{\psi_t}$ by using the so-called *$\hbar$-Fourier transform* of ψ_t:

$$\mathcal{F}_\hbar \psi_t(\xi) := \widehat{\psi}_t(\xi) := \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-ix\xi/\hbar}\psi_t(x)dx. \tag{1.2.6}$$

The $\hbar$-Fourier transform $\mathcal{F}_\hbar$ is an isometry of $L^2(\mathbf{R}^3)$, and if $\psi_t \in H^1(\mathbf{R}^3)$ one has the relation:

$$\mathcal{F}_\hbar\left(\frac{\hbar}{i}\nabla\psi_t\right)(\xi) = \xi \mathcal{F}_\hbar \psi_t(\xi).$$

As a consequence, (1.2.5) can be rewritten as

$$\langle \xi \rangle_{\psi_t} = \langle \xi\widehat{\psi}_t, \widehat{\psi}_t \rangle_{L^2(\mathbf{R}^3)} \tag{1.2.7}$$

so that it assumes a form more similar to (1.2.1).

Here we observe that the "wave-corpuscule" duality of a quantum particle is mathematically given by the correspondence between $\psi_t(x)$ and $\widehat{\psi}_t(\xi)$ via the $\hbar$-Fourier transform, in the sense that the average impulse of ψ_t equals the average position of $\widehat{\psi}_t$ and vice versa. Moreover, the classical quantities x and ξ can be considered in quantum mechanics, only via the two following operators:

$$x : \ \psi \mapsto x\psi$$

for the position, and

$$\hbar D_x : \ \psi \mapsto \frac{\hbar}{i}\nabla\psi$$

for the impulse. Note that these two operators are symmetric with respect to the L^2-scalar product, and are actually self-adjoint with respective domains $\mathcal{F}_\hbar(H^1(\mathbf{R}^3))$ and $H^1(\mathbf{R}^3)$. More generally, any (not necessarily bounded) self-adjoint operator on $L^2(\mathbf{R}^3)$ is called a *quantum observable*.

Now, the situation is the following: we have been able to associate with the classical observable x the quantum observable $\psi \mapsto x\psi$, and with the classical observable ξ the quantum observable $\hbar D_x$. Now a natural question arises: whether such a correspondence between classical observables and quantum observables can be generalized. Namely, given a classical observable $a(x,\xi)$, is there a natural way to associate with it a quantum observable (that could reasonably be denoted $a(x,\hbar D_x)$). This is precisely one of the purposes of the pseudodifferential calculus, which also provides an algebraic correspondence between the space of classical observables endowed with the usual multiplication, and the space of quantum observables endowed with the composition of operators.

Of course, there are cases where this correspondence can be demonstrated in a very simple way: if $a = a(x)$ does not depend on ξ, then the natural associated quantum observable is just the multiplication by the function a: $\psi \mapsto a\psi$. Similarly, using the fact that $\hbar D_x = \mathcal{F}_\hbar^{-1}\xi\mathcal{F}_\hbar$ (where ξ denotes the operator of multiplication by ξ), in the case when $a = a(\xi)$ does not depend on x, the associated quantum observable is $a(\hbar D_x) := \mathcal{F}_\hbar^{-1}a(\xi)\mathcal{F}_\hbar$. Note that this last definition is consistent with the usual one when $a(\xi)$ is a polynomial in ξ (in which case one obtains a differential operator).

Examples -

1. The *kinetic energy* $\dfrac{1}{2m}\xi^2$ gives rise to the quantum observable $-\dfrac{\hbar^2}{2m}\Delta$ where $\Delta = \sum_{j=1}^3 \partial_{x_j}^2$ is the Laplace operator ;

2. The total energy $\dfrac{1}{2m}\xi^2 + V(x)$ is associated to $H = -\dfrac{\hbar^2}{2m}\Delta + V(x)$, which is the celebrated *Schrödinger operator*.

Since the physical phenomena essentially consist in exchanges of energy, the study of the operator H is of particular interest in quantum mechanics.

The possible energies of a quantum particle submitted to the electric potential V are by definition the eigenvalues of H considered as an operator acting on $L^2(\mathbf{R}^3)$.

Quantum evolution -

In classical mechanics, Newton's law (1.1.1) permits us to predict the evolution of a particle once we know its initial position and momentum. The quantum counterpart can be derived by again using the analogy with Optics. From (1.2.2) we get

$$i\frac{\partial\varphi}{\partial t} = \omega\varphi$$

and by the so-called Planck-Einstein formula (initially obtained experimentally for light, but then generalized to matter by De Broglie in his theory of Matter Waves: see e.g. [Mes]), the energy of the wave is given by

$$E = h\nu = \hbar\omega.$$

As a consequence, φ satisfies:

$$i\hbar\frac{\partial\varphi}{\partial t} = E\varphi. \tag{1.2.8}$$

Since in quantum mechanics the energy is represented by the operator H, it becomes natural in view of (1.2.8) to require that the evolution of a quantum state is given by the equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\varphi \tag{1.2.9}$$

which is called the *Schrödinger equation.*

Until now we have dealt with a single quantum particle only. In the case when several (say N) particles are involved in the system that one wants to study, all the previous considerations can be easily generalized by instead denoting $x = (x^1, ..., x^N) \in \mathbf{R}^{3N}$ for the set of all the particles, and taking into account that the potential $V(x)$ must be the sum of the external electric field plus all the interactions between the various particles.

1.3 Semiclassical Analysis

As one can guess, the mathematical study of the Schrödinger operator H can be very difficult in general, depending on the potential V which is involved. As a consequence, one would like to dispose of any kind of approximation physically reasonable, allowing us to predict (at least qualitatively) many quantum properties of a system. Moreover, since classical mechanics (which in many aspects is much than quantum mechanics) describes very well (up to some accuracy) most of the common elementary physical phenomena, it is reasonable to hope that quantum mechanics is a kind of generalization of classical mechanics, in the sense that one should be able to recover the classical properties of a system by making some approximation of its quantum properties.

A general principle exists which permits us to give an answer to the two previous questions: the so-called *Bohr correspondence principle* asserts that classical mechanics is nothing but the limit as h tends to 0 of quantum mechanics.

Although this statement remains rather vague, it appears that in many instances it can be both specified and verified. The mathematical branch in which this is performed is commonly called *semiclassical analysis*, and its task consists principally in studying the spectral properties of H asymptotically as $h \to 0$ (that is for $h > 0$ small enough, without worrying about its actual physical value). However, due to its asymptotic character, semiclassical analysis also allows us to prove mathematically some typical quantum properties, which would be much more difficult to show by taking h fixed (and which are annihilated when the limit $h \to 0$ is taken).

An essential tool of semiclassical analysis consists of the use of the so-called *pseudodifferential calculus*, that we develop in the next sections. Actually, it may be applied in many other fields such as: the study of the singularities of solutions of partial differential equations (for which it has been initially developed: see e.g. [AlGe, Be1, ChPi, Ho2, KoNi, Sj1, Tr]); the Born-Oppenheimer approximation (used to study the quantum properties of molecules with heavy nuclei: [KMSW]); adiabatic theory (which studies slow varying systems, [Ju, Mar2]); solid state physics ([GRT, GMS]); scattering theory ([En, SiSo]) and many other aspects of spectral theory or else ([CdV, DiSj, Fo, GrSj, He1, He2, Iv, Mas, Ro, Sh, Ta], ...). Let us stress that in most of these fields h *is not* the Plank constant, but may represent different physical quantities such as: the inverse of the square root of the nuclear mass (in the Born-Oppenheimer approximation); the adiabatic parameter (in adia-

batic theory); the magnetic field strength (in solid state physics); the inverse of the square root of the energy (in high-energy spectral problems); or even the inverse of the norm of the position (in scattering theory).

1.4 About History

The history of microlocal analysis goes back more than forty years, and since a large number of mathematicians have contributed to the subject, it is quite difficult to describe with precision all the lines of its development. However, the interested reader may find a rich source of information in the historical notes that follow each chapter of the series of books [Ho2] by L. Hörmander. Here we mention that most of the motivation of our notes comes from techniques developed by J. Sjöstrand in [Sj1], and one of our purposes is to give a simplified and unified presentation of them. This has been made possible due to our observation that in a global context, the exponential microlocal estimates that we introduce in Chapter 3 (and that are quite elementary to prove) permit us to get rid of the rather heavy and difficult *pseudodifferential calculus in the complex domain* introduced in [Sj1]. Doing this, we believe that the main ideas of the proofs in [Sj1] appear in a more enlightened and clearer context which should allow a better appreciation and understanding of them.

Chapter 2

Semiclassical Pseudodifferential Calculus

2.1 Motivations and Notations

As we have already explained, one of the main motivations of the pseudodifferential calculus is to get an algebraic correspondence between the classical observables and the quantum observables (one calls it a *quantization* of the classical observables). In particular, this would permit to localize (within the limits allowed by the uncertainty principle) both in position and momentum variables any quantum state ψ: take a smooth cut-off $\chi = \chi(x,\xi) \in C_0^\infty(\mathbf{R}^{2n})$, then its associated quantum observable $\chi(x,\hbar D_x)$ applied to ψ will have the effect of (essentially) cutting off the cartesian product $\mathrm{Supp}\psi \times \mathrm{Supp}\widehat{\psi}$ outside $\mathrm{Supp}\chi$.

Another important feature of this calculus will consist in inverting the so-called *elliptic operators*: if $a(x,\xi)$ is a classical observable which never vanishes (and therefore is invertible in the multiplicative algebra of smooth functions), one would like to be able to invert also its quantization $a(x,\hbar D_x)$. This procedure (called the *construction of a parametrix*) will be possible when a satisfies a little bit more: namely that it is an invertible element of a special kind of subalgebras of $C^\infty(\mathbf{R}^n)$, called *spaces of symbols* (see the next section).

Many other properties are satisfied by the pseudodifferential operators, and we shall certainly not be exhaustive in these lectures, our purpose being to make understand how the things work and to show examples where they can be used. However there is another application that we only want to mention here,

and which is at the center of a whole field of interest in semiclassical analysis around the so-called *Weyl formula*: the pseudodifferential calculus permits to approximate some spectral projectors associated to the Schrödinger operator. We refer to the excellent book of Shubin [Sh] for a detailed approach of this problem.

Now, let us fix some standard notations that will be used all along this book.

If $x = (x_1, ..., x_n)$ denotes the current point of $\mathbf{R}^n$, we set:

$$D_x = \frac{1}{i}\frac{\partial}{\partial x} = \frac{1}{i}\nabla_x = \frac{1}{i}\partial_x = \frac{1}{i}(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$$

$$x^2 = x_1^2 + ... + x_n^2 \quad ; \quad |x| = \sqrt{x^2}$$

and, for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{N}^n$:

$$\begin{aligned} |\alpha| &= \alpha_1 + ... + \alpha_n \\ \alpha! &= (\alpha_1!)...(\alpha_n!) \\ x^\alpha &= x_1^{\alpha_1}...x_n^{\alpha_n} \\ D_x^\alpha &= D_{x_1}^{\alpha_1}...D_{x_n}^{\alpha_n} \\ \partial_x^\alpha &= \partial_{x_1}^{\alpha_1}...\partial_{x_n}^{\alpha_n} \end{aligned}$$

(although the two notations $|x|$ and $|\alpha|$ are incoherent since $\mathbf{N}^n \subset \mathbf{R}^n$, we use them because they are very traditional, and actually their meaning will always be clear from the context). We also denote $xy = x \cdot y = \langle x, y \rangle := x_1y_1 + ... + x_ny_n$ the standard scalar product between the two vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ of $\mathbf{R}^n$. Finally, we recall the very useful multidimensional Leibniz formula, valid for any C^∞ functions f, g on $\mathbf{R}^n$ and for any $\alpha \in \mathbf{N}^n$:

$$\partial_x^\alpha(fg) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!}(\partial_x^\beta f)(\partial_x^{\alpha-\beta} g) \tag{2.1.1}$$

where $\beta \leq \alpha$ means by definition that $\beta_j \leq \alpha_j$ for all $j \in \{1, ..., n\}$.

2.2 Spaces of Symbols

Roughly speaking, what we call *symbols* from now on is what we have called *classical observables* until now. In these lectures we concentrate on some restricted classes of symbols, in the double sense that they will always be globally

defined, and that they will satisfy estimates of a special kind. More general estimates could also be treated, as well as locally defined symbols, but the fact is that most of the difficulties of the theory are already encountered with our simpler classes, so that the reader will in any case learn all the basic techniques of microlocal analysis, without having to care about additional problems which could have the effect of obscuring the concepts (which in some sense remain always relatively simple below the technical discussion). For a very general presentation of the theory, one may e.g. consult the very complete series of books of Hörmander [Ho2].

Let $g \in C^\infty(\mathbf{R}^d; \mathbf{R}^*_+)$ (the space of C^∞ functions on $\mathbf{R}^d$ with values in $\mathbf{R}^*_+ = (0, \infty)$) satisfying for any $\alpha \in \mathbf{N}^d$

$$\partial_x^\alpha g = \mathcal{O}(g) \tag{2.2.1}$$

uniformly on $\mathbf{R}^d$. Such a function is called an *order function* on $\mathbf{R}^d$, and the simplest examples are given by

$$\langle x' \rangle^m := (1 + |x'|^2)^{m/2}$$

where $m \in \mathbf{R}$ is fixed and $x' = (x_1, ..., x_k)$ with $k \leq d$. Other examples are functions such as $e^{\alpha \langle x \rangle}$ with $\alpha \in \mathbf{R}$, or more generally $e^{f(x)}$ where f is smooth and bounded together with all its derivatives. However, a function such as e^{x^2} is not an order function, and neither is any function greater than it (see exercise n.1 at the end of this chapter).

Note that although we have denoted $x \in \mathbf{R}^d$ the variable of g, in practice we shall almost always have $d = 2n$ and x replaced by (x, ξ) representing the position-momentum variables. In fact, we shall sometimes also deal with $d = 3n$ and x replaced by (x, y, ξ) where the extra variable y plays the role of an integrated variable, in a similar way as when one expresses the distribution kernel of an operator as a function of (x, y).

A first property of this notion of order functions is:

Proposition 2.2.1 *If g is an order function on $\mathbf{R}^d$, then so is the function $\frac{1}{g}$.*

Proof - This is an easy consequence of the Leibniz formula (2.1.1). Indeed, one has to show that for any $\alpha \in \mathbf{N}^d$, $\partial^\alpha(1/g) = \mathcal{O}(1/g)$. Denoting $\tilde{g} = 1/g$

and using the Leibniz formula to differentiate α times the identity $\tilde{g}g = 1$, the required estimate is easily obtained by induction on $|\alpha|$. $\diamond$

For such an order function g we define the semiclassical space of symbols $S_d(g)$ by:

Definition 2.2.2 *A function $a = a(x;h)$ defined on $\mathbf{R}^d \times (0, h_0]$ for some $h_0 > 0$ is said to be in $S_d(g)$ if and only if a depends smoothly on x and for any $\alpha \in \mathbf{N}^d$, one has:*

$$\partial_x^\alpha a(x;h) = \mathcal{O}(g(x)) \tag{2.2.2}$$

uniformly with respect to $(x,h) \in \mathbf{R}^d \times (0, h_0]$.

In particular, $S_d(1)$ is the set of families of C^∞ functions on $\mathbf{R}^d$ parameterized by some $h \in (0, h_0]$, which are uniformly bounded together with all their derivatives.

Examples -

. Any $\chi \in C_0^\infty(\mathbf{R}^d)$ (the space of compactly-supported C^∞ functions on $\mathbf{R}^d$) is in $S_d(1)$;

. If $V = V(x) \in S_n(1)$, then the function $\xi^2 + V(x)$ is in $S_{2n}(\langle \xi \rangle^2)$. Note that it corresponds to the total energy (with mass $1/2$) defined in Section 1.2 ;

. For any $m \in \mathbf{R}$, $\langle x \rangle^m \in S_d(\langle x \rangle^m)$;

. The function $\mathbf{R}^{2n} \ni X = (x, \xi) \mapsto e^{ix.\xi}$ does not belong to $S_{2n}(\langle X \rangle^m)$ for any $m \in \mathbf{R}$, but belongs to any $S_{2n}(e^{\varepsilon \langle X \rangle})$ with $\varepsilon > 0$.

. The functions $e^{-x^2/h}$ and e^{x^2} do not belong to $S_n(g)$ for any order function g on $\mathbf{R}^n$ (just take the value at $x = 0$ of the Laplacian of the first one, and see exercise n.1 of this chapter for the second one).

Note that by Proposition 2.2.1 and Leibniz formula, we have the equivalence:

$$a \in S_d(g) \Leftrightarrow \frac{a}{g} \in S_d(1). \tag{2.2.3}$$

We endow $S_d(g)$ with the topology associated to the family of semi-norms $N_\alpha(a) = \mathrm{Sup}|\partial^\alpha a|$, and it can be verified easily that this makes $S_d(g)$ a Frechet space. The basic algebraic properties of the spaces $S_d(g)$ are the following ones.

Proposition 2.2.3 *Let g_1 and g_2 be two order functions on $\mathbf{R}^d$, and let $a \in S_d(g_1)$, $b \in S_d(g_2)$. Then $g_1 g_2$ is also an order function, and $ab \in S_d(g_1 g_2)$.*

Proof - This is an obvious consequence of Leibniz formula. ◇

Defining the ellipticity as:

Definition 2.2.4 *A symbol $a \in S_d(g)$ is said to be* **elliptic** *if there exists a positive constant C_0 such that*

$$|a| \geq \frac{1}{C_0} g$$

uniformly on $\mathbf{R}^d \times (0, h_0]$ (for some $h_0 > 0$),

then we have:

Proposition 2.2.5 *If $a \in S_d(g)$ is elliptic, then $\dfrac{1}{a} \in S_d\left(\dfrac{1}{g}\right)$.*

Proof - Denote $b = \dfrac{1}{a}$. Then the result is obtained by differentiating iteratively the relation $ab = 1$ and by using Leibniz formula. ◇

2.3 Semiclassical Expansions of Symbols

In this section we try to specify a little bit more the way in which the symbols may depend on the semiclassical parameter h. First of all we define the notation $\sim$ (the so-called *asymptotic equivalence* of symbols) that will be used very often in the sequels. All along this section, g denotes an arbitrary order function on $\mathbf{R}^d$.

Definition 2.3.1 *Let $a \in S_d(g)$ and $(a_j)_{j \in \mathbf{N}}$ a sequence of symbols of $S_d(g)$. Then we say that a is* **asymptotically equivalent** *to the formal sum $\sum_{j=0}^{\infty} h^j a_j$ in $S_d(g)$, and we denote*

$$a \sim \sum_{j=0}^{\infty} h^j a_j$$

if and only if for any $N \in \mathbf{N}$ *and for any* $\alpha \in \mathbf{N}^d$ *there exist* $h_{N,\alpha} > 0$ *and* $C_{N,\alpha} > 0$ *such that*

$$\left| \partial^\alpha \left(a - \sum_{j=0}^{N} h^j a_j \right) \right| \leq C_{N,\alpha} h^N g$$

uniformly on $\mathbf{R}^d \times (0, h_{N,\alpha}]$.

In other words, for any $N > 0$ the symbol a can be approximated by $\sum_{j=0}^{N} h^j a_j$ up to a symbol which vanishes together with all its derivatives as h^N when h goes to zero. In practice, the existence of $h_{N,\alpha}$ will not be explicitly written, being referred as to "for h small enough" at the end of an estimate. It will then be implicit that the estimate is valid for h in an interval of the form $(0, h_0]$ where h_0 depends of all the fixed parameters.

In the particular case when all the a_j's are identically zero, we denote:

$$a = \mathcal{O}(h^\infty) \text{ in } S_d(g) \quad \text{iff} \quad a \sim 0 \text{ in } S_d(g).$$

An important and surprising feature is that, although a series of the type $\sum_{j=0}^{\infty} h^j a_j$ has no reason to be convergent, one can always find a symbol which is asymptotically equivalent to it:

Proposition 2.3.2 *Let* $(a_j)_{j \in \mathbf{N}}$ *be an arbitrary sequence of symbols of* $S_d(g)$. *Then there exists* $a \in S_d(g)$ *such that* $a \sim \sum_{j=0}^{\infty} h^j a_j$ *in* $S_d(g)$. *Moreover,* a *is unique up to* $\mathcal{O}(h^\infty)$, *in the sense that the difference of two such symbols is* $\mathcal{O}(h^\infty)$ *in* $S_d(g)$. *Such a symbol* a *is called a* **resummation** *of the* **formal symbol** $\sum_{j \geq 0} h^j a_j$.

Proof - First of all, dividing everything by g and using (2.2.3), we can assume without loss of generality that $g = 1$.

Since the unicity up to $\mathcal{O}(h^\infty)$ is obvious, we concentrate on the existence of a. Then let $\chi \in C_0^\infty(\mathbf{R})$ be such that $\mathrm{Supp}\chi \subset [-2, 2]$, $\chi = 1$ on $[-1, 1]$. We have:

Lemma 2.3.3 *There exists a decreasing sequence of positive numbers* $(\varepsilon_j)_{j\in\mathbf{N}}$ *converging to zero, such that for any* $j \in \mathbf{N}$ *and* $\alpha \in \mathbf{N}^d$ *with* $|\alpha| \leq j$*, one has:*

$$\mathrm{Sup}_{x\in\mathbf{R}^d} \left| \left(1 - \chi(\frac{\varepsilon_j}{h})\right) \partial^\alpha a_j(x;h) \right| \leq h^{-1}$$

for h *small enough.*

Proof - Denoting

$$C_j = \mathrm{Sup}_{|\alpha|\leq j} \mathrm{Sup}_{x\in\mathbf{R}^d} |\partial^\alpha a_j(x;h)|$$

and using the fact that $\left(1 - \chi(\frac{\varepsilon_j}{h})\right)$ is non zero only for $h \leq \varepsilon_j$, we have:

$$h\mathrm{Sup}_{x\in\mathbf{R}^d} \left| \left(1 - \chi(\frac{\varepsilon_j}{h})\right) \partial^\alpha a_j(x;h) \right| \leq C_j\varepsilon_j \leq 1$$

if one has chosen the decreasing sequence $(\varepsilon_j)_{j\geq 0}$ in such a way that $\varepsilon_j \leq \dfrac{1}{C_j}$ for all $j \geq 0$ (one can take e.g. $\varepsilon_j = \min\{(k + C_k)^{-1} \ ; \ k \leq j\}$). $\diamond$

We then set:

$$a(x;h) = \sum_{j\geq 0} h^j \left(1 - \chi(\frac{\varepsilon_j}{h})\right) a_j(x;h)$$

where actually the sum contains only a finite number (depending on $h > 0$ fixed) of non zero terms (since $\varepsilon_j < h$ if j becomes too large). Thus a is a smooth function of $x \in \mathbf{R}^d$, and for any $\alpha \in \mathbf{N}^d$ one has:

$$|\partial^\alpha a(x;h)| \leq \sum_{j\leq|\alpha|} h^j |\partial^\alpha a_j(x;h)| + \sum_{j>|\alpha|} h^j \left| \left(1 - \chi(\frac{\varepsilon_j}{h})\right) \partial^\alpha a_j(x;h) \right|$$

and therefore, using Lemma 2.3.3:

$$|\partial^\alpha a(x;h)| \leq C_\alpha + \sum_{j>|\alpha|} h^{j-1} \leq C'_\alpha$$

where C_α and C'_α are positive constants.

Thus $a \in S_d(1)$, and for any $\alpha \in \mathbf{N}^d$ and $N \geq |\alpha|$ one has:

$$\left| \partial^\alpha \left(a - \sum_{j=0}^{N} h^j a_j \right) \right| \leq \sum_{j=0}^{N} h^j \left| \chi(\frac{\varepsilon_j}{h}) \partial^\alpha a_j \right| + \sum_{j\geq N+1} h^j \left| \left(1 - \chi(\frac{\varepsilon_j}{h})\right) \partial^\alpha a_j \right| .$$

Using again Lemma 2.3.3, we get:

$$\left|\partial^\alpha\left(a-\sum_{j=0}^{N}h^j a_j\right)\right| \leq \sum_{j=0}^{N} h^{N+j}\varepsilon_j^{-N}\left|(\frac{\varepsilon_j}{h})^N\chi(\frac{\varepsilon_j}{h})\right| C_{j,\alpha} + \sum_{j\geq N+1} h^{j-1}$$

where the $C_{j,\alpha}$'s are positive constants. Since the function $\mathbf{R}\ni t\mapsto t^N\chi(t)$ is bounded, we deduce easily from the estimate above that there exists a constant C_N such that for any $h>0$ sufficiently small:

$$\left|\partial^\alpha\left(a-\sum_{j=0}^{N}h^j a_j\right)\right| \leq C_N h^N$$

which completes the proof of Proposition 2.3.2. ◇

Remark 2.3.4 One can generalize the previous notion of equivalence by replacing h^j everywhere it appears by h^{m_j}, where $m_j\in\mathbf{R}$ satisfies: $m_j\to+\infty$ as $j\to+\infty$. Then one can prove in the same way an analogous result of resummation.

Remark 2.3.5 In the previous proof, we never used the fact that the a_j's are globally defined on $\mathbf{R}^d$. Indeed, a corresponding result for locally defined symbols is easy to state and to verify.

Application: WKB Solutions for the One Dimensional Schrödinger Operator - Let $V\in C^\infty(\mathbf{R}\,;\mathbf{R})$ and $E\in\mathbf{R}$, and let $x_0\in\mathbf{R}$ be such that $V(x_0)<E$. Then, for x close enough to x_0, one can consider the two smooth functions:

$$\varphi_\pm(x)=\pm\int_{x_0}^{x}\sqrt{E-V(y)}dy$$

which both satisfy $(\varphi'_\pm)^2=E-V$ (the so-called *eikonal equation*). In particular, $\varphi'_\pm\neq 0$ near x_0 and thus $\sqrt{|\varphi'_\pm|}$ is smooth there. Then, for any $a=a(x)$ smooth near x_0, one has:

$$\begin{aligned}
&\left(-h^2\frac{d^2}{dx^2}+V-E\right)\left(ae^{i\varphi_\pm/h}\right)\\
&\quad=\left[-2iha'\varphi'_\pm-iha\varphi''_\pm-h^2a''+\left((\varphi'_\pm)^2+V-E\right)a\right]e^{i\varphi_\pm/h}\\
&\quad=-ih\left[2a'\varphi'_\pm+a\varphi''_\pm-iha''\right]e^{i\varphi_\pm/h}\\
&\quad=-2ih\sqrt{\varphi'_\pm}\left[\left(a\sqrt{\varphi'_\pm}\right)'-ih\frac{a''}{2\sqrt{\varphi'_\pm}}\right]e^{i\varphi_\pm/h}
\end{aligned}$$